



Multi-Agent Systems with Reciprocal Interaction Laws

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Multi-Agent Systems with Reciprocal Interaction Laws

A dissertation presented

by

Xudong Chen

to

The Department of Engineering and Applied Sciences

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

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Computer Science

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Abstract

In this thesis, we investigate a special class of multi-agent systems, which we call reciprocal multi-agent (RMA) systems. The evolution of agents in a RMA system is governed by interactions between pairs of agents. Each interaction is reciprocal, and the magnitude of attraction/repulsion depends only on distances between agents. We investigate the class of RMA systems from four perspectives, these are two basic properties of the dynamical system, one formula for computing the Morse indices/co-indices of critical formations, and one formation control model as a variation of the class of RMA systems. An important aspect about RMA systems is that there is an equivariant potential function associated with each RMA system so that the equations of motion of agents are actually a gradient flow. The two basic properties about this class of gradient systems we will investigate are about the convergence of the gradient flow, and about the question whether the associated potential function is generically an equivariant Morse function. We develop systematic approaches for studying these two problems, and establish important results. A RMA system often has multiple critical formations and in general, these are hard to locate. So in this thesis, we consider a special class of RMA systems whereby there is a geometric characterization for each critical formation. A formula associated with the characterization is developed for computing the Morse index/co-index of each critical formation. This formula has a potential impact on the design and control of RMA systems. In this thesis, we also consider a formation control model whereby the control of formation is achieved by varying interactions between selected pairs of agents. This model can be interpreted in different ways in terms of patterns of information flow, and we establish results about the controllability of this control system for both centralized and decentralized problems.

Contents

Acknowledgements	vi
Dedication	vii
Introduction	1
1 Swarm Aggregation	7
1.1 Definitions and main theorem	8
1.2 Existence of solution and bounded size of equilibria	11
1.3 Clustering agents in dilute configurations	20
1.4 Dissipation zone	27
1.4.1 Definitions, theorem 1.4.1 and theorem 1.4.2	27
1.4.2 A lower bound for escaping velocity	29
1.4.3 A lower bound for escaping time	32
1.4.4 Proofs of theorem 1.4.1 and theorem 1.4.2	34
1.5 Asymptotic behavior of a semi-diverging gradient flow	35
1.5.1 Definitions and theorem 1.5.1	35
1.5.2 Metric property of clustering	36
1.5.3 Metric property of configuration	38
1.5.4 Proof of theorem 1.5.1	40
1.6 Convergence of the gradient flow	41
2 Index/Co-index Formula	44
2.1 Definitions and main theorem	45
2.2 The canonical partition of edges associated with an equilibrium	51
2.2.1 The canonical partition associated with an arbitrary configuration	51
2.2.2 The canonical partition associated with an equilibrium	54
2.2.3 Maximality of the canonical partition	57
2.3 Perturbing one sub-configuration while preserving shapes of others	59
2.3.1 Theorem 2.3.1	59
2.3.2 Proof of theorem 2.3.1	60
2.3.3 The infinitesimal version of the displacements	62
2.4 Hyperbolicity of a critical orbit	64
2.4.1 Theorem 2.4.1	64
2.4.2 The Hessian of the potential Ψ at an equilibrium p	65

2.4.3	The Hessian of a sub-potential Ψ_i at a sub-configuration p_i	67
2.4.4	Proof of the “only if” part of theorem 2.4.1	68
2.5	The index/co-index formula	72
3	Genericity of Equivariant Morse Functions	79
3.1	Definitions and main theorem	80
3.2	Graded lattice of edges	83
3.3	The Y -stratification of the configuration space	85
3.4	Perturbability of the gradient vector field	90
3.4.1	An upper bound for $\xi(p)$	92
3.4.2	Infinitesimal rigidity of a nondegenerate configuration	94
3.4.3	A lower bound for $\xi(p)$	96
3.5	The local lattice and the local stratification	98
3.5.1	The local lattice \mathcal{T}_p	99
3.5.2	The local stratification	101
3.6	Local genericity on the space of nondegenerate configurations	103
3.6.1	Theorem 3.6.1	103
3.6.2	Use of the frontier condition	104
3.6.3	Use of the fact that p is infinitesimally rigid	105
3.6.4	Use of the lower bound for $\xi(p)$	107
3.6.5	Proof of theorem 3.6.1	110
3.7	Proof of main theorem	111
4	Formation Control with Controllable Interactions	112
4.1	Definitions and main theorems	113
4.2	Matrix Lie algebra of interaction matrices	116
4.3	Approximate path-controllability on the space of nondegenerate configurations	121
4.4	Formation control using artificial potential functions	126
4.4.1	Centralized formation control	126
4.4.2	Decentralized formation control	127
	Conclusion	130
	Bibliography	135

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Dedicated to my mother, Lingdi Li, to my father, Jiale Chen, and to my fiancée, Fiona Hu.

Introduction

Over the last two decades, multi-agent system has been one of the most-studied topics in control theory. Numerous control models have been proposed for various purposes, among which, there is an important class of multi-agent systems, as we call in this thesis, the class of **Reciprocal Multi-Agent Systems**, or in short **RMA** systems.

Each RMA system is defined by a connected, undirected graph together with a family of interaction laws, as we will now describe in detail. Let $\Gamma = (V, E)$ be the graph with $V := \{1, \dots, N\}$ the set of vertices and E the set of edges. Consider the motion of a set of N agents in a purely kinematic model whereby agent \vec{x}_i feels the presence of \vec{x}_j if and only if they are adjacent, i.e, there is an edge between i and j in graph Γ , and they interact with each other reciprocally through an effect depending only on the pairwise distance between them. The equations of motion of agents $\vec{x}_1, \dots, \vec{x}_N \in \mathbb{R}^n$ with $N > n$ take the form

$$\dot{\vec{x}}_i = \sum_{j \in V(i)} g_{ij}(d_{ij})(\vec{x}_j - \vec{x}_i), \quad i = 1, \dots, N \quad (1)$$

The summation is over all vertices that are adjacent to vertex i . Each $g_{ij} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous differentiable function depending only on the mutual distance between \vec{x}_i and \vec{x}_j . We require g_{ij} be identical with g_{ji} for all $ij \in E$, in other words, interactions between agents are reciprocal.

The class of RMA systems has been investigated under various assumptions and from various perspectives. It has many applications in the control and design of unmanned autonomous vehicles (UAV's), swarming and flocking, etc. Questions concerning the level of interactions that is necessary for organizing such systems, questions about stability, robustness and etc. have all been treated to some degree.

We here give some examples of existing works related to the class of RMA systems. In work [1], Gazi and Passino worked on a problem about swarm aggregation under the assumption that Γ is a complete graph and the assumption that all interaction functions g_{ij} are identical, they specified a special class of interaction laws to achieve swarm aggregation and estimate the size of equilibria. In work [2], Anderson and Helmke focussed on a formation control problem with a particular family

of interaction laws, in particular, they apply equivariant Morse theory to establish lower bounds for number of critical formations. In work [3], Marshall and Broucke worked with a specific class of connected graphs with circulant connectivity, and established sufficient and necessary conditions for preserving cyclic group symmetry in planar configurations. In work [4], Olfati-Saber and Murray worked on consensus problems with switching graph topology and time delay under the assumption that each g_{ij} , $ij \in E$, is a constant function. We refer readers to [5–15] for more examples of works related to the class of RMA systems.

An important property about the class of RMA systems, as we will see in chapter 1, is that the equations we adopt to describe the evolution of configurations can be thought of as a gradient flow. The importance of gradient descent for finding solutions is widely appreciated both in mathematics and in the real world. Added to the usual reasons, in the context of formation control it can be used to provide a decentralized solution for a problem involving an arbitrary collection of agents. The novelty of this thesis lies not in the use of this technique but rather in describing some of its strengths and limitations in the the context of multi-agent systems. In this thesis, we will investigate the class of RMA systems from four perspectives. We will now give a brief summary, as well as motivations behind each of the perspectives.

1. Swarm aggregation. In chapter 1, we develop, among other things, a basic property about the class of RMA systems. We show that if Γ is connected, and if each interaction function g_{ij} , $ij \in E$, has fading attraction and infinite repulsion at zero separation, then system (1) evolves as a gradient flow and the equilibria associated with the gradient flow have bounded size. Moreover, we show that for each initial condition, the gradient flow converges to the set of equilibria. Conventional techniques for proving convergence of gradient flows on non-compact manifolds are, for example, constructing a Lyapunov function and then applying LaSalle’s principle. Yet, such technique may not work in our case under the highly realistic assumption that interactions functions have fading attractions. So in chapter 1, we will develop a new approach for studying swarm aggregation. This approach is a fusion of combinatorics and dynamical system. In particular, we will introduce a parametrized clustering which induces a partial order that reflects the granularity of partition of agents. This is a rich question

relating to classic techniques such as k -mean algorithm and its variants, and is useful for studying other multi-agent system problems concerning about large-size configurations.

2. The index/co-index formula. A potential function associated with a RMA system often has saddle points and multiple local minima and hence, the stable equilibria will divide the configuration space into disjoint regions of attraction. Knowledge about the location of each stable equilibrium and the knowledge about how regions of attractions partition the underlying configuration space will be valuable. In general, even counting the total number of stable equilibria is hard. (see, for example, [2, 7, 8] for details about this kind of counting problem.) In chapter 2, we consider a special class of RMA systems associated with a particular type of Laman graphs. We show, in this case, for each equilibrium associated with system (1), there is a geometric decomposition of the equilibrium into union of sub-configurations such that each sub-configuration is a line formation and is itself an equilibrium. We then establish the index/co-index formula which relates the Morse index/co-index of an equilibrium, as an algebraic term, to this geometric decomposition. We show that the Morse index/co-index of the equilibrium is the sum over Morse indices/co-indices of these decomposed sub-configurations. This formula has a potential impact on the design and control of multi-agent systems as it makes possible for us to locate or place critical formations with various Morse indices/co-indices over the configuration space.

3. Genericity of equivariant Morse functions. We observe that system (1) depends only on relative distances between agents, implying that it is just the shape of the configuration and not the Euclidean coordinates of the individual agents that matters. Naturally there is a group action of special Euclidean group $SE(n)$ acting on a configuration, emphasizing the invariance of rigid motion, so each RMA system is actually an equivariant gradient system with respect to the Lie group $SE(n)$. The associated potential function is said to be an equivariant Morse function, if there are only finitely many critical orbits, and the Hessian of the potential is nondegenerate when restricted to the normal bundle of each critical orbit. This basic condition is often assumed in the analysis of RMA systems, and is an indispensable assumption in the study of many problems, including problems of counting critical formations, problems of computing indices and co-indices of critical formations and problems

of characterizing regions of attractions, etc. Yet, it is still an open question whether the potential function associated with system (1) is generically an equivariant Morse function. In chapter 3, we will investigate this open question under the assumption that Γ is a complete graph, and establish important results about the genericity. The analysis presents several challenges. In part, these challenges arise from the fact that the associated potential function is in a particular form, implying that choices for perturbing the resulting gradient vector fields are restricted. A systematic approach will be established for dealing with this restriction. This approach combines ideas from combinatorics, algebraic geometry and differential topology, etc., and has implications for other problems as well.

4. Formation control with controllable interaction laws. In chapter 4, we will investigate a formation control model as a variation of system (1). We replace each interaction function g_{ij} , $ij \in E$, by a scalar control u_{ij} . So if we let $V(i) := \{j \in V | ij \in E\}$ be the set of vertices adjacent to i , then the control model is

$$\dot{\vec{x}}_i = \sum_{j \in V(i)} u_{ij}(\vec{x}_j - \vec{x}_i), \quad i = 1, \dots, N \quad (2)$$

Each u_{ij} , $ij \in E$, controls interaction between \vec{x}_i and \vec{x}_j , and similarly, we require $u_{ij} = u_{ji}$, i.e, interaction between agents are reciprocal. We will investigate the formation control model from two perspectives. One is from the view of classic nonlinear control. We will show that the control system (2) is approximately path-controllable on a connected, open and dense subset of the configuration space. In particular, we will introduce the notion of interaction matrix, and compute the matrix Lie algebra associated with it, this computation lays the foundation for the proof of the approximate path-controllability. The other is from the view of decentralized formation control. There are different interpretations of the control model in terms of the pattern of information flow. For example, It matters how much information each agent can access to or collect from others, and if the interaction between \vec{x}_i and \vec{x}_j is under control, then it also matters whether \vec{x}_i observes \vec{x}_j and takes control of the interaction or vice versa. So besides an undirected graph that describes the pattern of interaction, a digraph is needed to describe the pattern of information flow. In this thesis, we will consider only a specific case with a particular pattern of information flow. A decentralized control law that relates

to an artificial potential function will be investigated in detail.

Chapter 1

Swarm Aggregation

1.1 Definitions and main theorem

In this chapter, we will investigate the problem of swarm aggregation in a class of reciprocal multi-agent (RMA) systems. Let $\Gamma = (V, E)$ be a connected, undirected graph with $V := \{1, \dots, N\}$ the set of vertices and E the set of edges. For each vertex i , we let $V(i) \subset V$ be the collection of vertices that are adjacent to vertex i . We recall that the equations of motion, for agents $\vec{x}_1, \dots, \vec{x}_N \in \mathbb{R}^n$ with $N > n$, take the form

$$\dot{\vec{x}}_i = \sum_{j \in V(i)} g_{ij}(d_{ij})(\vec{x}_j - \vec{x}_i), \quad i = 1, \dots, N \quad (1.1)$$

Each $g_{ij} : \mathbb{R}^+ \rightarrow \mathbb{R}$, as a function of distance between \vec{x}_i and \vec{x}_j , is continuous differentiable. We require $g_{ij} = g_{ji}$ for all $ij \in E$. In this thesis, we will impose two conditions on each g_{ij} , $ij \in E$, and they are:

Strong repulsion. $\lim_{d \rightarrow 0} dg_{ij}(d) = -\infty$ and $\lim_{d \rightarrow 0} \int_d^1 xg_{ij}(x)dx = -\infty$.

Fading attraction. $g_{ij}(d) > 0$ if $d \gg 1$ and $\lim_{d \rightarrow \infty} dg_{ij}(d) = 0$.

The term $dg_{ij}(d)$ appears because it represents the actual attraction/repulsion between \vec{x}_i and \vec{x}_j , and for convenience, we let

$$\bar{g}_{ij}(d) := dg_{ij}(d) \quad (1.2)$$

The conditions of *strong repulsion* and *fading attraction* about the interaction function somehow mimic natural laws. For example, we recall the Leonard-Jones potential between a pair of neutral atoms or molecules is given by

$$V_{LJ}(d) = 4\epsilon \left[\left(\frac{\sigma}{d} \right)^{12} - \left(\frac{\sigma}{d} \right)^6 \right] \quad (1.3)$$

where ϵ is the depth of the potential well, σ is the finite distance at which the inter-particle potential is zero. Consequently, the interaction between a pair of agents is given by

$$\frac{\partial V_{LJ}}{\partial d}(d) = 4\epsilon \left[-\frac{12}{d} \left(\frac{\sigma}{d} \right)^{12} + \frac{6}{d} \left(\frac{\sigma}{d} \right)^6 \right] \quad (1.4)$$

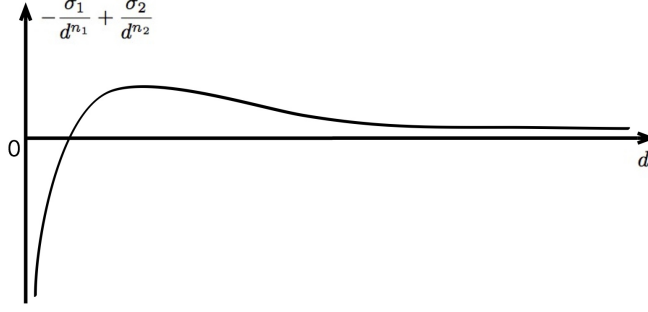


Figure 1: A class of rational functions that satisfy conditions of *strong repulsion* and *fading attraction*.

and we verify that this resulting interaction satisfies conditions of *strong repulsion* and *fading attraction*. In fact, it is not hard to see that for any two positive numbers $\sigma_1 > 0$ and $\sigma_2 > 0$, and for any two positive integers n_1 and n_2 with $n_1 > n_2 > 0$, the interaction function

$$\bar{g}_{ij}(d) := -\frac{\sigma_1}{d^{n_1}} + \frac{\sigma_2}{d^{n_2}} \quad (1.5)$$

satisfies conditions of *strong repulsion* and *fading attraction*, and we illustrate in figure 1 this special class of rational functions.

As interactions between pairs of agents are reciprocal, consequently the centroid of a configuration is invariant along the evolution, so we may assume in this chapter that the centroid of a configuration is located at the origin. We also observe that an interaction law g_{ij} produces infinite repulsion at zero separation. Thus equation (1.1) is only well defined if we agree to limit our attention to configurations in which \vec{x}_i doesn't collide with \vec{x}_j for each $ij \in E$. Thus our model should be thought of as being defined on an appropriate open subset of $\mathbb{R}^{n \times N}$. So the configuration space is defined by

$$P := \{(\vec{x}_1; \dots; \vec{x}_N) \in \mathbb{R}^{n \times N} \mid \sum_{i=1}^N \vec{x}_i = 0 \text{ and } \vec{x}_i \neq \vec{x}_j, \forall ij \in E\} \quad (1.6)$$

We observe that the equations we adopt to describe the evolution of the configuration are actually a gradient flow over P . The potential function associated is a symmetric function of the individual

agents and is given by

$$\Psi(\vec{x}_1, \dots, \vec{x}_N) := \sum_{ij \in E} \int_1^{d_{ij}} \bar{g}_{ij}(x) dx \quad (1.7)$$

In this chapter, we will develop, among other things, one basic property of this class of RMA systems.

Theorem 1.1 (Compact Global Attractor). *Each RMA system described by equation (1.1) is a gradient system over the configuration space P , the associated potential function is Ψ . Suppose the network topology $\Gamma = (V, E)$ is connected, and suppose each interaction function g_{ij} , $ij \in E$, satisfies conditions of strong repulsion and fading attraction. Then the set of equilibria associated with the gradient flow is compact, in particular, there exist two positive numbers l_- and l_+ such that the distance between any two adjacent agents in an equilibrium lies in the closed interval $[l_-, l_+]$. Moreover, for each initial condition, the gradient flow exists for all time and converges to the set of equilibria.*

We here note a work by Veysel Gazi and Kevin M. Passino. In [1], the authors considered the problem of swarm aggregation within a special class of RMA system whereby the network topology Γ is a complete graph, and all interaction functions between pairs of agents are identical. The class of attraction/repulsion functions, considered in their paper, includes interaction functions with bounded repulsion and linear attraction, interaction functions with unbounded repulsion and linearly bounded from below attraction, and interaction functions with unbounded repulsion and almost constant attraction. In any of such cases, a Lyapunov function was constructed to establish the result of swarm aggregation. Yet, this conventional technique can't be applied in our case by our assumption of fading attraction. So we will develop along the proof a new technique for the analysis of swarm aggregation.

The rest of this chapter is organized as follows. In section 1.2, we show that the resulting gradient flow of the RMA system is collision-free, so in particular, the solution of the dynamical system exists for all time. We will also establish important metric properties of the set of equilibria, we show that the size of each equilibrium is bounded both above and below. The rest of this chapter, from section 1.3 to section 1.6, are about the analysis of the convergence of the gradient flow. In section 1.3, we introduce a parametrized definition of clustering which induces a partial order that

reflects the granularity of the partition of agents, and we establish important properties of the lattice defined in this way. In section 1.4, we introduce the notion of dissipation zone which is a special co-dimensional one subspace that divides the configuration space into two disjoint, connected components. We establish a lower bound for the loss of potential whenever a gradient flow reaches a dissipation zone. In section 1.5, we introduce the notion of semi-diverging gradient flow, and investigate its path behavior by using the parametrized clustering. In section 1.6, we combine the results we established in the earlier sections to prove the convergence of the gradient flow.

1.2 Existence of solution and bounded size of equilibria

For each initial condition p , we let $\varphi_t(p)$ be the solution of the gradient flow at time t . If the solution exists for all time $t > 0$, we then simply denote by $\varphi_{\geq 0}(p)$ the entire gradient flow. Let a, b be two numbers with $0 \leq a \leq b \leq \infty$, we define a subset of P as

$$P_a^b := \{p \in P \mid a \leq d_{ij} \leq b, ij \in E\} \quad (1.8)$$

In this chapter, we will simply write P^b if $a = 0$ and P_a if $b = \infty$. Our goal in this section is to develop theorem 1.2.1 and theorem 1.2.2.

Theorem 1.2.1. *Consider the RMA system described by equation (1.1). For each initial condition $p \in P$, the solution of the gradient flow exists for all time. In fact, there exists a number $a > 0$ associated with p such that $\varphi_{\geq 0}(p) \subset P_a$.*

Theorem 1.2.2. *Consider the RMA system described by equation (1.1). There exist two positive numbers l_- and l_+ such that each equilibrium associated with the gradient flow is contained in $P_{l_-}^{l_+}$.*

we will now first prove theorem 1.2.1.

Proof of theorem 1.2.1. It suffices to show that there is a positive number a associated with the initial

condition p such that if $\varphi_t(p)$ exists, then $\varphi_t(p) \in P_a$. Let

$$\psi_0 := \min\left\{\int_1^d \bar{g}_{ij}(x)dx \mid d \in \mathbb{R}^+, ij \in E\right\} \quad (1.9)$$

By conditions of *strong repulsion* and *fading attraction*, we know ψ_0 exists. On the other hand, by

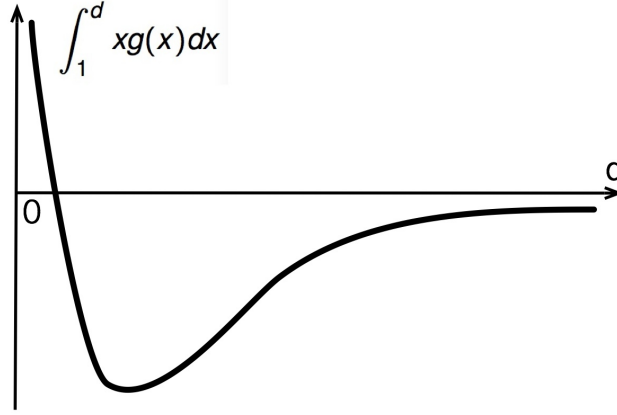


Figure 2: For each interaction function g that satisfies conditions of *strong repulsion* and *fading attraction*, the minimum of $\int_1^d xg(x)dx$ exists as d varies over \mathbb{R}^+ .

condition of *strong repulsion* alone, we know exists a positive number a such that for any $a' \in (0, a)$ and for any $ij \in E$, we have

$$\int_1^{a'} \bar{g}_{ij}(x)dx + (|E| - 1)\psi_0 > \Psi(p) \quad (1.10)$$

where $|E|$ is the size of the graph Γ . The potential $\Psi(\varphi_t(p))$, as a function of t , is non-increasing along the evolution, so inequality (1.10) implies that at any time $t > 0$, the distance between any two adjacent agents in $\varphi_t(p)$ is bounded below by a . This then establishes the theorem. \square

The rest of this section is devoted to the proof of theorem 1.2.2. We will prove the existence of upper bound l_+ and the existence of lower bound l_- in lemma 1.2.3 and corollary 1.2.5, respectively.

Before going on, we first define two positive numbers associated with the family of interaction

laws. Let α and β be defined so that

$$\begin{aligned} g_{ij}(d) &< 0, \quad \forall ij \in E \quad \& \quad \forall d \in (0, \alpha) \\ g_{ij}(d) &> 0, \quad \forall ij \in E \quad \& \quad \forall d \in (\beta, \infty) \end{aligned} \tag{1.11}$$

These two numbers exist by conditions of *strong repulsion* and *fading attraction*. In fact, let α_{ij} and β_{ij} be the first and last zero of g_{ij} , $ij \in E$, then we may just set

$$\begin{aligned} \alpha &= \min\{\alpha_{ij} | ij \in E\} \\ \beta &= \max\{\beta_{ij} | ij \in E\} \end{aligned} \tag{1.12}$$

We now state lemma 1.2.3.

Lemma 1.2.3. *Consider the RMA system described by equation (1.1). There is a positive number l_+ such that the distance between any two adjacent agents in an equilibrium is less than l_+ .*

Proof. The proof is done by contradiction. We assume that for any distance d , there is an equilibrium p with a pair of adjacent agents, say \vec{x}_1 and \vec{x}_N , such that $d_{1N} \geq d$. We now choose $d := N\beta$.

Let x_i^j be the j -th coordinate of agent \vec{x}_i . Rotate the configuration, if necessary, so that $\vec{x}_N - \vec{x}_1$ lies in the x^1 -axis, and we assume $x_1^1 < x_N^1$. To keep \vec{x}_1 balanced, there is at least one agent, say \vec{x}_2 , such that $\vec{x}_2^1 \leq \vec{x}_1^1 + \beta$ because otherwise, the interaction between \vec{x}_1 and \vec{x}_i for each $1i \in E$ is an attraction. So then, the projection of the interaction on agent \vec{x}_1 is positive along the x^1 -axis and hence, \vec{x}_1 can't be balanced.

Consequently, we have $d_{2N} \geq (N-1)\beta > \beta$, so by assumption $g_{2N}(d_{2N}) \geq 0$. This, in particular, implies that

$$g_{1N}(d_{1N})(x_N^1 - x_1^1) + g_{2N}(d_{2N})(x_N^1 - x_2^1) > 0 \tag{1.13}$$

In other words, agent \vec{x}_N attracts the two-agent sub-configuration formed by \vec{x}_1 and \vec{x}_2 as a whole,

along x^1 -axis. So there must exist an agent among \vec{x}_1 and \vec{x}_2 , say \vec{x}_2 , such that

$$g_{12}(d_{12})(x_1^1 - x_2^1) + g_{2N}(d_{2N})(x_N^1 - x_2^1) > 0 \quad (1.14)$$

Then to keep \vec{x}_2 balanced, there is at least one agent \vec{x}_3 such that $x_3^1 \leq x_2^1 + \beta \leq x_1^1 + 2\beta$.

Since $d_{3N} \geq (N - 2)\beta > \beta$, we have $g_{3N}(d_{3N}) \geq 0$. So \vec{x}_N attracts the three-agent sub-configuration formed by \vec{x}_1 , \vec{x}_2 and \vec{x}_3 as a whole, along x^1 -axis. Consequently there is an agent among \vec{x}_1 , \vec{x}_2 and \vec{x}_3 , say \vec{x}_3 , such that

$$\sum_{i=1}^2 g_{i3}(d_{i3})(x_i^1 - x_3^1) + g_{3N}(d_{3N})(x_N^1 - x_3^1) > 0 \quad (1.15)$$

Similarly, to keep \vec{x}_3 balanced, we locate agent \vec{x}_4 with $x_4^1 \leq x_3^1 + 3\beta$.

Repeat the process, we then get $x_k^1 \leq x_1^1 + (k - 1)\beta$ for each $k = 1, \dots, N - 1$. Consequently $d_{kN} > \beta$ for all $k \neq N$. But then,

$$\sum_{i=1}^{N-1} g_{iN}(d_{iN})(x_i^1 - x_N^1) < 0 \quad (1.16)$$

So, agent \vec{x}_N can't be balanced. This contradicts to the assumption that p is an equilibrium. \square

The existence of upper bound l_+ is now clear. We will now prove the existence of lower bound l_- . We start by defining a particular subset of the configuration space. Let ϵ and r be two positive numbers, let \vec{x}_0 be the center of \vec{x}_1 and \vec{x}_2 , i.e, $\vec{x}_0 := \frac{1}{2}(\vec{x}_1 + \vec{x}_2)$, and let d_{i0} be the Euclidean distance between \vec{x}_i and \vec{x}_0 . We then define a subset of P as

$$Z(\epsilon, r, N) := \left\{ p \in P \left| \begin{array}{ll} d_{ij} \geq d_{12} = \epsilon, & \forall ij \in E \\ d_{i0} \leq r, & \forall i < N \end{array} \right. \right\} \quad (1.17)$$

So for each configuration p in $Z(\epsilon, r, N)$, the distance d_{12} is minimal among all distances between pairs of adjacent agents in p , and all agents are contained in the closed ball of radius r centered at \vec{x}_0 .

Let $f(p)$ be the gradient vector field associated with Ψ , i.e,

$$f(p) := -\nabla \Psi(p) \quad (1.18)$$

and let $f_i(p)$ be the restriction of $f(p)$ to agent \vec{x}_i , i.e,

$$f_i(p) := \sum_{j \in V(i)} g_{ij}(d_{ij})(\vec{x}_j - \vec{x}_i) \quad (1.19)$$

Let \hat{p} be a sub-configuration of p formed by agents $\vec{x}_{i_1}, \dots, \vec{x}_{i_m}$ with $i_1 < \dots < i_m$. Similarly, we can define $\hat{f}(\hat{p})$ and $\hat{f}_{i_j}(\hat{p})$ as if \hat{p} were isolated. To be explicit, let $\hat{\Gamma}$ be a subgraph of Γ defined by restricting Γ to vertices i_1, \dots, i_m , and let

$$\hat{\Psi}(\vec{x}_{i_1}, \dots, \vec{x}_{i_m}) := \sum_{i_j i_k \in E} \int_1^{d_{i_j i_k}} \bar{g}_{i_j i_k}(x) dx \quad (1.20)$$

be a sub-potential of Ψ by restricting Ψ to the subgraph $\hat{\Gamma}$. We then define

$$\hat{f}(\hat{p}) := -\nabla \hat{\Psi}(\hat{p}) \quad (1.21)$$

and let $\hat{f}_{i_j}(\hat{p})$ be the restriction of $\hat{f}(\hat{p})$ to agent \vec{x}_{i_j} .

Lemma 1.2.4. *Let p be a configuration in $Z(\epsilon, r, N)$, and let \vec{e}_{0i} be a unit vector defined by*

$$\vec{e}_{0i} := \frac{\vec{x}_i - \vec{x}_0}{|\vec{x}_i - \vec{x}_0|} \quad (1.22)$$

Let $\langle \cdot, \cdot \rangle$ be the normal inner-product, so then the projection of $f_i(p)$ along \vec{e}_{0i} is $\langle f_i(p), \vec{e}_{0i} \rangle$. Let

$$\xi(p) := \max\{\langle f_i(p), \vec{e}_{0i} \rangle | i = 1, \dots, N\} \quad (1.23)$$

and let

$$\eta(\epsilon, r, N) := \inf\{\xi(p) | p \in Z(\epsilon, r, N)\} \quad (1.24)$$

If vertices 1 and 2 are adjacent in Γ , then

$$\lim_{\epsilon, r \rightarrow 0} \eta(\epsilon, r, N) = +\infty \quad (1.25)$$

The result holds for all $N \geq 2$, and it does't depend on how ϵ and r go to zero.

Proof. There are choices of (ϵ, r, N) such that $Z(\epsilon, r, N)$ is the empty set, then $\eta(\epsilon, r, N) = +\infty$. So it suffices to consider the case $Z(\epsilon, r, N)$ is nonempty. The proof is done by induction on the number of agents.

Base case. Consider the case where we have only two agents \vec{x}_1 and \vec{x}_2 . By condition of *strong repulsion* on g_{12} , we have

$$\lim_{\epsilon, r \rightarrow 0} \eta(\epsilon, r, 2) = \lim_{\epsilon \rightarrow 0} |\bar{g}_{12}(\epsilon)| = \infty \quad (1.26)$$

this establishes the base case.

Inductive step. We assume that the lemma holds for $N \leq k - 1$ and we prove for the case $N = k$. The proof is done by contradiction, i.e, we assume for any $\epsilon > 0$ and $r > 0$, there exist ϵ' and r' with $0 < \epsilon' \leq \epsilon$ and $0 < r' \leq r$, together with a configuration $p \in Z(\epsilon', r', k)$ such that $\xi(p)$ is bounded above by a fixed number $\xi_0 > 0$.

Since the graph Γ is connected, there is a chain of subgraphs

$$\emptyset = \Gamma_0 \subset \Gamma_1 \cdots \subset \Gamma_k = \Gamma \quad (1.27)$$

each Γ_i is a connected subgraph consisting exactly of i vertices. Relabel the vertices, if necessary, so that the set of vertices associated with Γ_i is $\{1, \dots, i\}$ for each $i = 1, \dots, k$.

Let \hat{p} be a sub-configuration of p formed by agents $\vec{x}_1, \dots, \vec{x}_{k-1}$. By induction, $\xi(\hat{p})$ can be made arbitrarily large by shrinking ϵ and r . So choose a positive number K , sufficiently large, such that $\xi(\hat{p}) = K\xi_0$. We may as well assume that ϵ and r are small enough so that all interactions among p are repulsions.

We may assume that $\xi(\hat{p})$ is achieved by agent \vec{x}_j , i.e, $\xi(\hat{p}) = \langle \hat{f}_j(\hat{p}), \vec{e}_{0j} \rangle$. Consider the projec-

tion of $f_j(p)$ along \vec{e}_{0j} , we have

$$\langle f_j(p), \vec{e}_{0j} \rangle = \xi(\hat{p}) + g_{jk}(d_{jk}) \langle \vec{x}_k - \vec{x}_j, \vec{e}_{0j} \rangle \quad (1.28)$$

So to keep $\langle f_j(p), \vec{e}_{0j} \rangle \leq \xi(p) \leq \xi_0$, we must have

$$g_{jk}(d_{jk}) \langle \vec{x}_j - \vec{x}_k, \vec{e}_{0j} \rangle \geq (K-1)\xi_0 \quad (1.29)$$

This, in particular, implies $\langle \vec{x}_k - \vec{x}_j, \vec{e}_{0j} \rangle > 0$. So then an important consequence about the geometry of the triangle formed by \vec{x}_0 , \vec{x}_j and \vec{x}_k is that $d_{k0} > d_{j0}$, and hence

$$\langle \vec{x}_k - \vec{x}_j, \vec{e}_{0j} \rangle < \langle \vec{x}_k - \vec{x}_j, \vec{e}_{0k} \rangle \quad (1.30)$$

So if \vec{x}_j is the only agent that is adjacent to \vec{x}_k , then

$$\langle f_k(p), \vec{e}_{0k} \rangle = g_{jk}(d_{jk}) \langle \vec{x}_j - \vec{x}_k, \vec{e}_{0k} \rangle > g_{jk}(d_{jk}) \langle \vec{x}_j - \vec{x}_k, \vec{e}_{0j} \rangle > (K-1)\xi_0 \quad (1.31)$$

which is a contradiction because K can be made arbitrarily large.

So there is at least one agent, say \vec{x}_{i_1} , other than \vec{x}_j such that the two agents \vec{x}_{i_1} and \vec{x}_k are adjacent. In addition, we have

$$g_{i_1k}(d_{i_1k}) \langle \vec{x}_k - \vec{x}_{i_1}, \vec{e}_{0k} \rangle > \frac{1}{k} [g_{jk}(d_{jk}) \langle \vec{x}_j - \vec{x}_k, \vec{e}_{0k} \rangle - \xi_0] \geq \frac{K-2}{k} \xi_0 \quad (1.32)$$

because otherwise

$$\langle f_k(p), \vec{e}_{0k} \rangle \geq g_{jk}(d_{jk}) \langle \vec{x}_j - \vec{x}_k, \vec{e}_{0k} \rangle + \sum_{i \neq j, i \neq k} g_{ik}(d_{ik}) \langle \vec{x}_i - \vec{x}_k, \vec{e}_{0k} \rangle > \xi_0 \quad (1.33)$$

We may as well assume $K > 2$, inequality (1.32) then implies $\langle \vec{x}_{i_1} - \vec{x}_k, \vec{e}_{0k} \rangle > 0$. So again, for the

triangle formed by \vec{x}_0 , \vec{x}_k and \vec{x}_{i_1} we have $d_{i_1 0} > d_{k0}$, and hence

$$\langle \vec{x}_{i_1} - \vec{x}_k, \vec{e}_{0k} \rangle < \langle \vec{x}_{i_1} - \vec{x}_k, \vec{e}_{0i_1} \rangle \quad (1.34)$$

We now make a change of variable by $K' := (K - 2)/k + 1$, so then by combining inequality (1.32) and inequality (1.34)

$$g_{i_1 k}(d_{i_1 k}) \langle \vec{x}_k - \vec{x}_{i_1}, \vec{e}_{0i_1} \rangle > g_{i_1 k}(d_{i_1 k}) \langle \vec{x}_k - \vec{x}_{i_1}, \vec{e}_{0k} \rangle > (K' - 1)\xi_0 \quad (1.35)$$

By identifying inequality (1.35) with inequality (1.29), we then conclude that there is an agent \vec{x}_{i_2} adjacent to \vec{x}_{i_1} such that

$$g_{i_2 i_1}(d_{i_2 i_1}) \langle \vec{x}_{i_1} - \vec{x}_{i_2}, \vec{e}_{0i_1} \rangle > \frac{K' - 2}{k} \xi_0 \quad (1.36)$$

So again $\langle \vec{x}_{i_2} - \vec{x}_{i_1}, \vec{e}_{0i_1} \rangle > 0$ and hence, $d_{i_2 0} > d_{i_1 0}$.

Repeat the process, we then get a sequence of agents $\vec{x}_{i_1}, \vec{x}_{i_2}, \dots$ with $d_{i_{k+1} 0} > d_{i_k 0}$ for all $k = 1, 2, \dots$. This sequence must be infinite because K can be made arbitrarily large. This then contradicts to the fact that there are only finite number of agents. \square

The next corollary establishes the existence of lower bound l_- .

Corollary 1.2.5. *Consider the RMA system described by equation (1.1). There is a positive number l_- such that the distance between any two adjacent agents in an equilibrium is greater than l_- .*

Proof. The proof is done by contradiction, i.e, we assume that for any $\epsilon > 0$, there is an equilibrium p with two adjacent agents, say \vec{x}_1 and \vec{x}_2 , such that $d_{12} \leq \epsilon$. We may as well assume that d_{12} is the minimum distance between adjacent agents.

Let ζ_0 be a positive number, by condition of *strong repulsion*, we can choose ϵ such that $|\bar{g}_{ij}(d)| > \zeta_0$ for any $d \in (0, \epsilon)$ and for any $ij \in E$. Let

$$\rho_1 := \sup\{d \in \mathbb{R}^+ \mid |\bar{g}_{ij}(d)| = \zeta_0/N \text{ for some } ij \in E\} \quad (1.37)$$

we assume ζ_0 is large enough so that $\rho_1 < \alpha$. Then to keep \vec{x}_1 balanced, there is at least one agent, say \vec{x}_3 , with $d_{13} \leq \rho_1$ because otherwise,

$$\left| \sum_{i=3}^N g_{1i}(d_{1i})(\vec{x}_i - \vec{x}_1) \right| < \sum_{i=3}^N |\bar{g}_{1i}(d_{1i})| < \sum_{i=3}^N |\bar{g}_{1i}(\rho_1)| < \zeta_0 < |\bar{g}_{12}(d_{12})| \quad (1.38)$$

and hence, agent \vec{x}_1 can't be balanced. Let \vec{x}_0 be the center of \vec{x}_1 and \vec{x}_2 , then the three agents \vec{x}_1 , \vec{x}_2 and \vec{x}_3 are contained in the open ball $B_{r_1}(\vec{x}_0)$ centered at \vec{x}_0 with radius $r_1 := \rho_1 + \epsilon$. Notice that both ϵ and ρ_1 go to zero as ζ_0 goes to infinity, so r_1 can be made arbitrarily small by increasing ζ_0 .

Let \hat{p} be the sub-configuration formed by agents \vec{x}_1 , \vec{x}_2 and \vec{x}_3 , and let $\xi(\hat{p})$ be defined by equation (1.23) as if \hat{p} were isolated from other agents. Then by lemma 1.2.4, we can make $\xi(\hat{p})$ arbitrarily large by shrinking both ϵ and r_1 . Let $\zeta_1 := \xi(\hat{p})$, and let

$$\rho_2 := \sup\{d \in \mathbb{R}^+ \mid |\bar{g}_{ij}(d)| = \zeta_1/N \text{ for some } ij \in E\} \quad (1.39)$$

By lemma 1.2.4, we can make ζ_1 arbitrarily large by shrinking r_1 . On the other hand, we know

$$\lim_{\zeta_0 \rightarrow \infty} r_1 = 0 \quad (1.40)$$

and hence

$$\lim_{\zeta_0 \rightarrow \infty} \zeta_1 = \infty \quad (1.41)$$

So we can again assume that ζ_1 is sufficiently large so that $\rho_2 < \alpha$. Suppose ζ_1 is achieved by agent \vec{x}_3 of \hat{p} , i.e, $\zeta_1 = \langle f_3(\hat{p}), \vec{e}_{03} \rangle$, then to keep \vec{x}_3 balanced, we have to place at least one agent, say \vec{x}_4 , around \vec{x}_3 such that $d_{34} < \rho_2$. In other words, the four agents $\vec{x}_1, \dots, \vec{x}_4$ are contained in the open ball $B_{r_2}(\vec{x}_0)$ with $r_2 := \epsilon + \rho_1 + \rho_2$, and r_2 can be made arbitrarily small by increasing ζ_1 . By the same reason, we conclude that

$$\lim_{\zeta_0 \rightarrow \infty} r_2 = 0 \quad (1.42)$$

Repeat the process, we then successively locate $\vec{x}_5, \dots, \vec{x}_N$. All of the agents are contained in

an open ball $B_r(\vec{x}_0)$. Moreover, the radius r approaches to zero as ζ_0 goes to infinity. By lemma 1.2.4, we can make r sufficiently small so that $\xi(p) > 0$. This contradicts to the assumption that p is an equilibrium. \square

By combining lemma 1.2.3 and corollary 1.2.5, we then prove theorem 1.2.2. Notice that the set P_{l-}^{l+} is compact, so the set of equilibria, as a closed subset in a compact set, is also compact. In the rest of this chapter, we will show that each gradient flow converges to the set of equilibria. In other words, the set of equilibria is a global compact attractor of the gradient system.

1.3 Clustering agents in dilute configurations

A parametrized **clustering** $\sigma(l, \epsilon)$ with parameters $l > 0$ and $\epsilon > 0$ on a configuration p is a

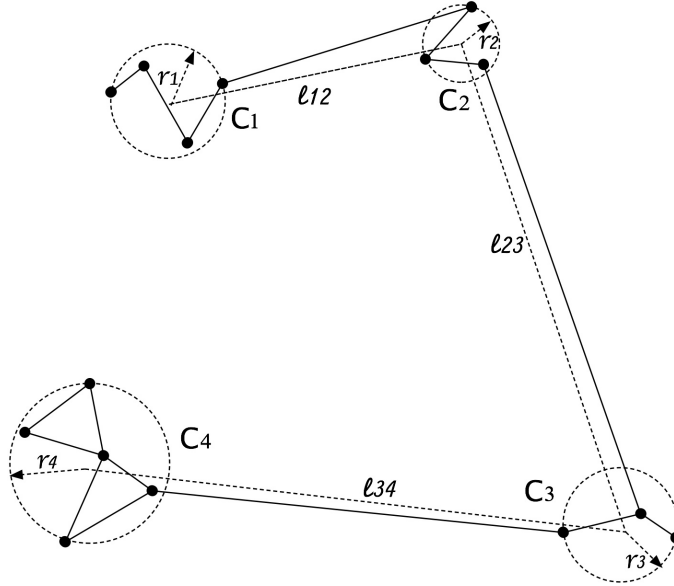


Figure 3: A clustering $\sigma(l, \epsilon)$ on a configuration that satisfies three conditions. 1) each subgraph Γ_i defined by restricting Γ to cluster C_i is connected, 2) each adjacent-cluster distance l_{ij} is greater than l and 3) the ratio of the radius of a cluster C_i to any of its adjacent-cluster distance l_{ij} is less than ϵ .

partition of agents of p into disjoint union of clusters C_1, \dots, C_M . The partition has to satisfy three conditions:

1. Let Γ_i be a subgraph of Γ associated with C_i , i.e, if $C_i = \{\vec{x}_{i_1}, \dots, \vec{x}_{i_k}\}$, then Γ_i is the restriction of Γ to vertices i_1, \dots, i_k . We require each Γ_i be connected.
2. Let C_i and C_j be two **adjacent clusters**, i.e, there is an agent \vec{x}_i in C_i and an agent \vec{x}_j in C_j such that the two agents are adjacent. Let l_{ij} be an **adjacent-cluster distance**, i.e, l_{ij} is the distance between centers of two adjacent clusters C_i and C_j . We then require l_{ij} be greater than l .
3. Let r_i be the radius of cluster C_i , then $r_i/l_{ij} < \epsilon$ for all j with C_j adjacent to C_i .

Any configuration admits the **trivial clustering**, namely the one with only one cluster containing all the agents. If a configuration admits a nontrivial clustering, then its size is bounded below. Conversely we show any configuration with sufficiently large radius will admit a nontrivial clustering.

Lemma 1.3.1 (Nontrivial clustering). *Given a pair of positive parameters (l, ϵ) , there exists a number $r(l, \epsilon) > 0$ such that if the radius of a configuration is greater than $r(l, \epsilon)$, then the configuration admits a nontrivial clustering $\sigma(l, \epsilon)$.*

Proof. Suppose not, then given any $r > 0$, there exists a configuration p with its radius greater than r , but only admits the trivial clustering. For such a configuration, there must exist at least a pair of adjacent agents \vec{x}_i and \vec{x}_j such that $d_{ij} \leq l$, otherwise the agent-wise clustering will be admitted by p and is nontrivial.

Divide N agents in p into disjoint groups by the following rule: two agents \vec{x}_i and \vec{x}_j are in the same group if and only if there is a chain $\vec{x}_{a_1}, \dots, \vec{x}_{a_m}$ with $a_1 = i$, $a_m = j$ such that \vec{x}_{a_q} and $\vec{x}_{a_{q+1}}$ are adjacent and $d_{a_q a_{q+1}} \leq l$, and this holds for all $q = 1, \dots, m-1$. This rule uniquely determines the partition.

Let G_1, \dots, G_k be the groups associated with the partition. The radius of each group is less than $\frac{1}{2}Nl$, so we may assume that there is more than one group because the radius of p can be arbitrarily large. Then by the same reason, there exist at least a pair of adjacent groups G_i and G_j such that the distance between their centers is less than Nl/ϵ .

Integrate G_1, \dots, G_k by the following rule: two groups G_i and G_j will be integrated if and only if there is a chain $G_{a_1}, \dots, G_{a_{m'}}$ with $a_1 = i$, $a_{m'} = j$ such that G_{a_q} and $G_{a_{q+1}}$ are adjacent, and the distance between centers of G_{a_q} and $G_{a_{q+1}}$ is less than Nl/ϵ , this holds for all $q = 1, \dots, m' - 1$.

Let $G'_1, \dots, G'_{k'}$ be groups of agents after integration. The radius of each G'_i is less than $\frac{1}{2\epsilon}N^2l$. The number of groups will strictly decrease after each step of integration, so $k' < k$. If $k' > 1$, then we can repeat the process to integrate groups into even larger ones by applying the same rule. In finite steps, there remains only one group that contains all the agents, and its radius is bounded above by $\frac{1}{2\epsilon}N^Nl$. This is a contradiction because we can choose radius of p arbitrarily large. \square

A configuration may admit multiple clusterings with respect to the same pair of parameters. We then ask whether there is a canonical one among all qualified clustering. A clustering $\sigma(l, \epsilon)$ induces a partition on the set of vertices V . Let (V_1, \dots, V_k) be the family of disjoint, nonempty subsets with respect to $\sigma(l, \epsilon)$, then each V_i collects indices of agents in cluster C_1 . Then there is a partial order on clusterings describing the granularity of the partition. Suppose $\sigma'(l, \epsilon)$ is another clustering and induces a different partition $(V'_1, \dots, V'_{k'})$ on V . We say $\sigma(l, \epsilon)$ is a **refinement** of σ' and denote by $\sigma(l, \epsilon) \succ \sigma'(l, \epsilon)$ if $k > k'$, and each V_i is a subset of V'_j for some j .

Lemma 1.3.2 (Linear order of clusterings with fixed parameters). *Given a pair of parameters (l, ϵ) with $\epsilon < 1/4$, any two different clustering $\sigma(l, \epsilon)$ and $\sigma'(l, \epsilon)$ is comparable and hence, all clusterings with fixed parameter (l, ϵ) form a linearly ordered set, i.e., $\sigma_1(l, \epsilon) \succ \dots \succ \sigma_n(l, \epsilon)$ with $\sigma_n(l, \epsilon)$ the trivial clustering.*

Proof. Suppose there are two non-comparable clusterings $\sigma(l, \epsilon) = (C_1, \dots, C_k)$, $\sigma'(l, \epsilon) = (C'_1, \dots, C'_{k'})$ on p . Without loss of generality, we assume

1. $\vec{x}_1 \in C_1$ and $\vec{x}_2 \in C_2$
2. $\vec{x}_1, \vec{x}_2 \in C'_1$
3. $\vec{x}_3 \in C_1 \cap C'_2$ and \vec{x}_3 is adjacent to \vec{x}_1

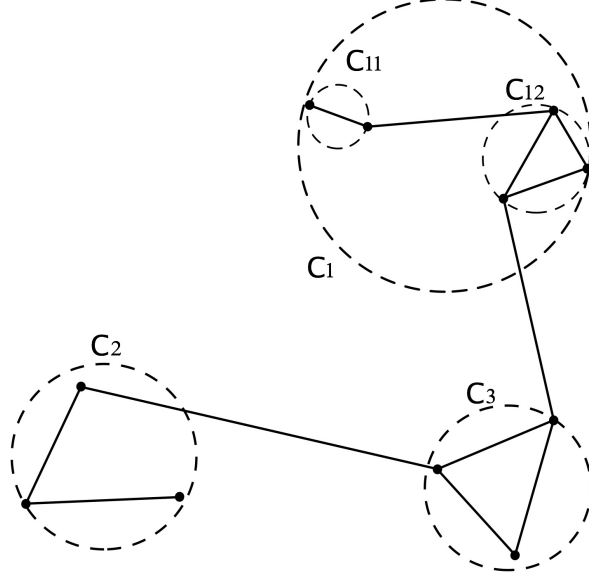


Figure 4: A refinement of clustering by dividing C_1 into two sub-clusters as C_{11} and C_{12} .

Let r_1, r_2 and r'_1 be the radii of clusters C_1, C_2 and C'_1 , respectively. Let $r := \max\{r_1, r_2\}$, then $r > 0$ because C_1 contains at least two agents \vec{x}_1 and \vec{x}_3 . Notice that $d_{12} > (1/\epsilon - 2)r$ since \vec{x}_1 and \vec{x}_2 are in different clusters with respect to $\sigma(l, \epsilon)$, while $d_{12} < 2r'_1$ as the two agents are both contained in C'_1 . On the other hand, $d_{13} > (1/\epsilon - 2)r'_1$ because \vec{x}_1 and \vec{x}_3 are in different clusters with respect to $\sigma'(l, \epsilon)$. So then $\frac{1}{2}(\frac{1}{\epsilon} - 2)^2 r \leq d_{13} \leq 2r$ which is a contradiction if $\epsilon < 1/4$. \square

Remark. In the rest of this section, we will implicitly assume $\epsilon < 1/4$. The clustering $\sigma_1(l, \epsilon)$ is then indecomposable, and can be regarded as the **canonical clustering** with respect to (l, ϵ) .

Let $\sigma(l, \epsilon)$ and $\sigma'(l', \epsilon')$ be two clusterings on p with respect to parameters (l, ϵ) and (l', ϵ') respectively, if the two clusterings induce the same partition on V , we will then simply write $\sigma(l, \epsilon) \simeq \sigma'(l', \epsilon')$.

Lemma 1.3.3 (Variation of clusterings with varying parameters). *Let (l, ϵ) and (l', ϵ') be two pairs of parameters with $l' \geq l$ and $\epsilon' \leq \epsilon$. Let $\{\sigma_i(l, \epsilon)\}_{i=1}^n$ and $\{\sigma'_i(l', \epsilon')\}_{i=1}^{n'}$ be the linearly ordered sets of clusterings with respect to (l, ϵ) and (l', ϵ') respectively, then for each $k = 1, \dots, n'$, we have*

$\sigma'_k(l', \epsilon') \simeq \sigma_j(l, \epsilon)$ for some $j = 1, \dots, n$.

Proof. Suppose (C_1, \dots, C_k) is a partition with respect to $\sigma'(l', \epsilon')$, so then each subgraph Γ_i associated with C_i is connected, and by assumption if C_i and C_j are adjacent, then

$$\begin{aligned} l_{ij} &> l' \geq l \\ r_i/l_{ij} &< \epsilon' \leq \epsilon \end{aligned} \tag{1.43}$$

So then the clustering $\sigma(l, \epsilon) := (C_1, \dots, C_k)$ satisfies all three defining conditions. \square

Clustering is useful for investigating gradient flows of dilute configurations, and we now state the main theorem of this section.

Theorem 1.3.4 (Consistent clustering on a diverging gradient flow). *If there were an initial condition p such that along the gradient flow $\varphi_{\geq 0}(p)$ the radius of configuration can't be bounded above, then there would be a monotone sequence of times $\{t_i\}_{i \in \mathbb{N}}$ approaching to infinity such that*

1. *each configuration in the sequence $\{\varphi_{t_i}(p)\}_{i \in \mathbb{N}}$ admits a nontrivial clustering $\sigma(l_i, \epsilon)$ that will induce the same partition on the set of vertices V .*
2. *if we partition the agents in $\varphi_{t_i}(p)$ into disjoint clusters with respect to $\sigma(l_i, \epsilon)$, then the minimum adjacent-cluster distance approaches to infinity along the sequence.*
3. *there is a fixed number $R > 0$ that bounds any radius of a cluster in a configuration in the sequence.*

Proof. As the radius of configuration along the flow curve isn't bounded above, so there is a time sequence $\{t_i\}_{i \in \mathbb{N}}$ approaching to infinity such that the radius of the configuration $\varphi_{t_i}(p)$ monotonically increases and approaches to infinity along the sequence. For convenience, we let p_i denote $\varphi_{t_i}(p)$.

Let (l, ϵ) be a pair of positive parameters, since the radius of the configuration diverges along the sequence, by lemma 1.3.1, there is an index $n_{l, \epsilon}$ associated with (l, ϵ) such that for each configuration p_i , $i \geq n_{l, \epsilon}$, in the sequence, the canonical clustering $\sigma(l, \epsilon)$ on p_i is nontrivial. Fix ϵ and let $\{l_i\}_{i \in \mathbb{N}}$

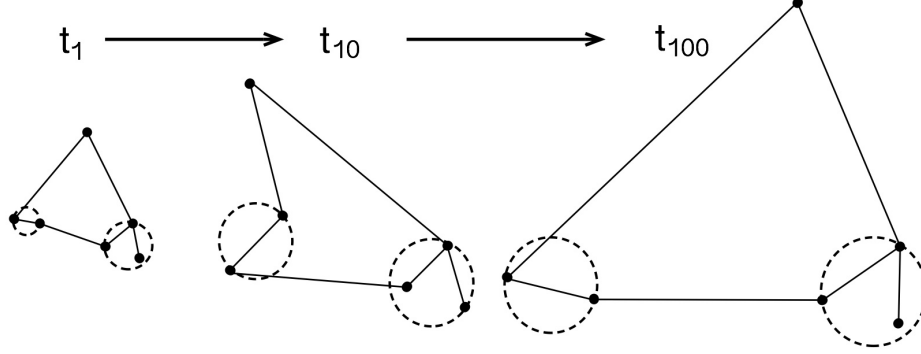


Figure 5: Consistent clustering on a diverging sequences of configurations that satisfies three conditions. 1) consistent classification of agents, 2) bounded-size of clusters and 3) diverging adjacent-cluster distances.

be a positive monotonically increasing sequence approaching to infinity, we then get the sequence of indices $\{n_{l_i, \epsilon}\}_{i \in \mathbb{N}}$ correspondingly.

Pick a subsequence $\{p_{j_i}\}_{i \in \mathbb{N}}$ out of $\{p_i\}_{i \in \mathbb{N}}$, and the index j_i , $i \in \mathbb{N}$, satisfies

$$\begin{cases} j_1 > n_{l_1, \epsilon} \\ j_i > \max\{j_{i-1}, n_{l_i, \epsilon}\}, \quad i > 1 \end{cases} \quad (1.44)$$

so then, the minimum adjacent-cluster distance in p_{j_i} associated with $\sigma(l_{j_i}, \epsilon)$ will approach to infinity along the subsequence. For convenience, we may assume that the original sequence already satisfies this condition, and we will write $\{p_i, \sigma(l_i, \epsilon)\}_{i \in \mathbb{N}}$ to emphasize the associated clustering on each p_i .

There are only finitely many partitions on V , so there is a subsequence $\{p_{j_i}, \sigma(l_{j_i}, \epsilon)\}_{i \in \mathbb{N}}$ chosen in a way that each clustering $\sigma(l_{j_i}, \epsilon)$ on p_{j_i} induces the same partition on V . Again we assume that the original sequence has already satisfied this condition. So far we have constructed a sequence $\{p_i\}_{i \in \mathbb{N}}$ that satisfies the first two conditions in the statement of the theorem.

Last we show that there is a subsequence out of $\{p_i\}_{i \in \mathbb{N}}$ that will satisfy condition 3. The proof is done by induction on the number of agents.

Base case. The case $N = 2$ is trivially true.

Inductive step. We assume that the theorem holds for $N < k$ and prove for the case $N = k$. Along

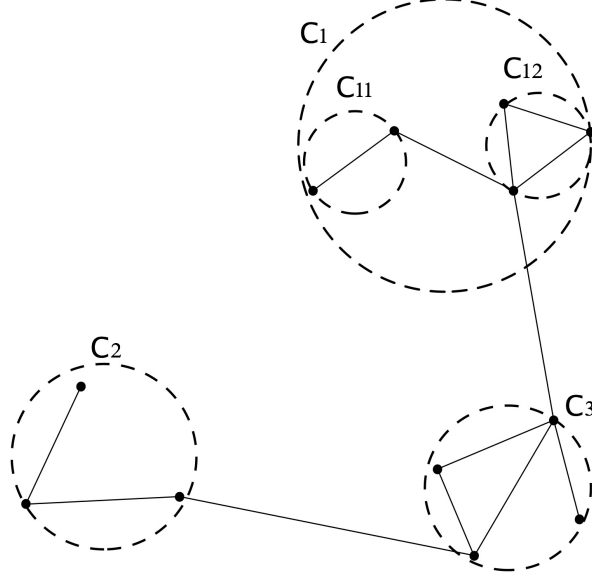


Figure 6: Clusters C_i , $1 \leq i \leq 3$, are w.r.t the canonical clustering $\sigma(l, \epsilon)$ on p , while the clusters C_{11} , C_{12} are w.r.t the canonical clustering $\hat{\sigma}(\hat{l}, \epsilon)$ on the sub-configuration formed by agents in C_1 . Let $\sigma(\hat{l}, \epsilon)$ be a clustering on p that agrees with $\hat{\sigma}(\hat{l}, \epsilon)$ inside C_1 , but agrees with $\sigma(l, \epsilon)$ outside. It suffices to show that the adjacent-cluster distance between C_i , $i \neq 1$, and C_{1j} , $j = 1, 2$, is greater than \hat{l} if C_i and C_{1j} are adjacent. This holds because by assumption C_i and C_1 are adjacent, so then $d(C_i, C_{1j}) > (l - r_1) > (1/\epsilon - 1)r_1 > (1/\epsilon - 1)\hat{l}/2 > 3/2\hat{l}$.

the sequence $\{p_i, \sigma(l_i, \epsilon)\}_{i \in \mathbb{N}}$, we may assume that there is at least one cluster of agents, say C_1 , with its radius approaching to infinity. For convenience, we denote by $\{\hat{p}_i\}_{i \in \mathbb{N}}$ the sequence of sub-configurations formed by agents in C_1 . The number of agents in each \hat{p}_i is fixed and less than k , so by induction there will be a subsequence $\{\hat{p}_{j_i}, \hat{\sigma}_{j_i}(\hat{l}_{j_i}, \epsilon)\}_{i \in \mathbb{N}}$, with each $\hat{\sigma}_{j_i}(\hat{l}_{j_i}, \epsilon)$ a clustering on \hat{p}_{j_i} , such that it satisfies all three conditions in the statement of the theorem. Since cluster C_1 is indecomposable with respect to $\sigma(l_{j_i}, \epsilon)$, it then implies that $\hat{l}_{j_i} < l_{j_i}$. So by lemma 1.3.3, there is a clustering $\sigma(\hat{l}_{j_i}, \epsilon)$ on p_{j_i} such that it agrees with $\hat{\sigma}(\hat{l}_{j_i}, \epsilon)$ inside C_1 , but with $\sigma(l_{j_i}, \epsilon)$ outside (also see figure 6 for more detail). Pick the subsequence $\{p_{j_i}, \sigma(\hat{l}_{j_i}, \epsilon)\}_{i \in \mathbb{N}}$ out of $\{p_i, \sigma(l_i, \epsilon)\}_{i \in \mathbb{N}}$ with $\sigma(l_{j_i}, \epsilon)$ replaced with $\sigma(\hat{l}_{j_i}, \epsilon)$. If there is any other diverging cluster in the subsequence, then we repeat the process. The whole procedure terminates in finite steps of repetition, and we get a sequence that satisfies all three conditions in the statement of the theorem. \square

1.4 Dissipation zone

1.4.1 Definitions, theorem 1.4.1 and theorem 1.4.2

Let $d > 0$ be a distance, let r be a radius and let ij be an edge in E . We then define two subsets of P by

$$X_{ij}(d) := \{p \in P | d_{ij} = d\} \quad (1.45)$$

$$Y(r) := \{p \in P | r_p = r\} \quad (1.46)$$

where r_p is the radius of a configuration p .

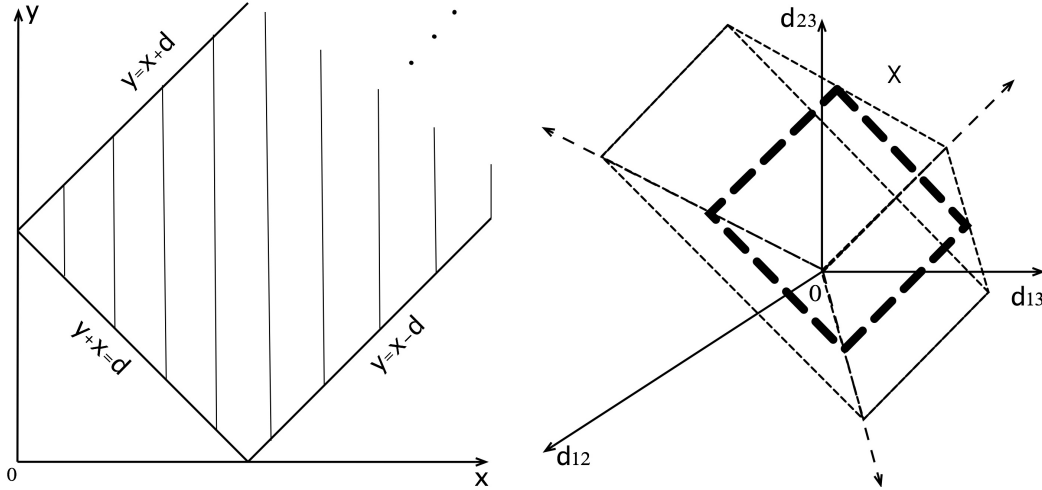


Figure 7: An example of $X_{12}(d)$ in the case where we have only three agents \vec{x}_1 , \vec{x}_2 and \vec{x}_3 . If we fix the distance between \vec{x}_2 and \vec{x}_2 as $d_{12} = d$, then all possible values of (d_{13}, d_{23}) are contained in the shaded region in the left diagram. The right diagram then shows how this region is embedded in the shape space using relative distances as coordinates. In particular, we see that X_{12} partitions the entire space into two connected components.

A distance d (or a radius r) is said to be **absent** in a configuration p if $d_{ij} \neq d$ for any $ij \in E$ (or $r(p) \neq r$), and $X_{ij}(d)$ (or $Y(r)$) is said to be a **dissipation zone** if the distance d (or the radius r) is absent in any equilibrium associated with the gradient flow.

Our goal in this section is to develop theorem 1.4.1 and theorem 1.4.2, these two theorems

establish positive lower bounds for the loss of potential of a gradient flow once it reaches a dissipation zone.

Theorem 1.4.1. *Let d be a distance absent in any equilibrium, and let*

$$\mu_{ij}(N, d) := \inf\{|f(p)| \mid p \in X_{ij}(d)\} \quad (1.47)$$

then $\mu_{ij}(N, d) > 0$. If along a gradient flow $\varphi_{\geq 0}(p)$, there is a moment $t > 0$ at which $\varphi_t(p) \in X_{ij}(d)$, then there is a $\tau_\mu > 0$ such that during period $[t, t + \tau_\mu]$, the loss of potential along the gradient flow is at least $\mu_{ij}^2(N, d)\tau_\mu/4$.

There is a similar version of theorem 1.4.1 by replacing $X_{ij}(d)$ with $Y(r)$, as we state below.

Theorem 1.4.2. *Let r be a radius absent in any equilibrium, let*

$$\nu(N, r) := \inf\{|f(p)| \mid r_p = r\} \quad (1.48)$$

then $\nu(N, r) > 0$. If along a gradient flow $\varphi_{\geq 0}(p)$, there is a moment $t > 0$ at which $\varphi_t(p) \in Y(r)$, then there is a $\tau_\nu > 0$ such that during period $[t, t + \tau_\nu]$, the loss of potential along the gradient flow is at least $\nu^2(N, d)\tau_\nu/4$.

In this section, we will mainly focus on the development of theorem 1.4.1, but all the arguments along the development can be used to prove theorem 1.4.2. A complete proof will be given at the end of this section.

Let I be a closed neighborhood of d , we then let $X_{ij}(I) := \{X_{ij}(d) \mid d \in I\}$. Notice that

$$\frac{d}{dt}\Psi(\varphi_t(p)) = -|f(\varphi_t(p))|^2 \quad (1.49)$$

So theorem 1.4.1 can be established if we can

1. find a closed neighborhood I of d such that $\inf\{|f(p)| \mid p \in X_{ij}(I)\} > \frac{1}{2}\mu_{ij}(N, d)$.

2. compute a lower bound for the period that a gradient flow takes to escape out of $X_{ij}(I)$ from $X_{ij}(d)$.

The rest of this section is then organized by this order, we will first prove the existence the interval I , and then estimate the escaping time.

1.4.2 A lower bound for escaping velocity

Our goal here is to establish theorem 1.4.3.

Theorem 1.4.3. *Let $d > 0$ be a distance absent in any equilibrium, and let $\mu_{ij}(N, d)$ be defined by equation (1.47), then $\mu_{ij}(N, d) > 0$. Moreover, as a function of x , $\mu_{ij}(N, x)$ is continuous at d . This, in particular, implies that there exists a closed neighborhood I of d with $\mu_{ij}(N, d') > \frac{1}{2}\mu_{ij}(N, d)$ for any $d' \in I$.*

The proof of theorem 1.4.3 will be given after lemma 1.4.4. We start by proving the existence of a positive lower bound for $|f(p)|$ as p varies over $X_{ij}(d)$.

Lemma 1.4.4. *Suppose $d > 0$ is absent in any equilibrium, then $\mu_{ij}(N, d) > 0$.*

Proof. As the edge ij is fixed during the proof, so for simplicity, we will simply write $X(\cdot)$ and $\mu(\cdot, \cdot)$ by omitting their subindices. Let a, b be two positive numbers with $0 \leq a \leq b \leq \infty$, and let $X_a^b(d) := X(d) \cap P_a^b$. Similarly, we will write X^b if $a = 0$ and write X_a if $b = +\infty$. Define

$$\begin{aligned}\mu_1(N, d) &:= \inf\{|f(p)| \mid p \in X_a^b(d)\} \\ \mu_2(N, d) &:= \inf\{|f(p)| \mid p \in X(d) - X_a(d)\} \\ \mu_3(N, d) &:= \inf\{|f(p)| \mid p \in X(d) - X^b(d)\}\end{aligned}\tag{1.50}$$

then

$$\mu(N, d) = \min\{\mu_1(N, d), \mu_2(N, d), \mu_3(N, d)\}\tag{1.51}$$

so it suffices to find a and b such that $\mu_i(N, d) > 0$ for each $i = 1, 2, 3$.

Proof that $\mu_1(N, d) > 0$. This holds for any two positive numbers a and b . Because the subset $X_a^b(d)$

is compact, and by assumption the gradient field doesn't vanish over $X_a^b(d)$.

Proof that $\mu_2(N, d) > 0$. Let δ be a positive number, we show that there exists $a > 0$ such that $|f(p)| > \delta$ for any $p \in X(d) - X_a(d)$. Choose a sufficiently small, we then apply the same arguments in the proof of theorem 1.2.2 to argue that if there are two adjacent agents in p such that their distance is less than a , then to keep $|f(p)|$ less than δ , all the other agents can be successively located in a small open ball. In particular, the radius of the ball approaches to zero as a goes to zero. So then by lemma 1.2.4, $|f(p)|$ can be made arbitrary large by shrinking a .

Proof that $\mu_3(N, d) > 0$. The proof is done by induction on the number of agents.

Base case. In the case $N = 2$, the set $X(d)$ is a singleton and

$$\mu(2, d) = \sqrt{2}|\bar{g}_{12}(d)| > 0 \quad (1.52)$$

Induction step. Assume that $\mu(N, d) > 0$ for $N \leq k - 1$ and we prove for the case $\mu_3(k, d) > 0$. Let

$$\hat{\mu}(k - 1, d) := \min\{\mu(m, d) | 2 \leq m \leq k - 1\} \quad (1.53)$$

Let (l, ϵ) be a pair of positive parameters for clustering. By lemma 1.3.1 and lemma 1.3.2, the canonical clustering on p is nontrivial if the radius of $p \in X(d)$ exceeds certain threshold. Let b be twice the threshold, we show that there is a positive lower bound for $|f(p)|$ as p varies over $X(d) - X^b(d)$.

Without loss of generality, we assume \vec{x}_1 and \vec{x}_2 are adjacent, and $d_{12} = d$. As the distance d is fixed, if we choose l large enough with ϵ fixed, then \vec{x}_1 and \vec{x}_2 are belong to the same cluster, say C_1 . Let \hat{p} be the sub-configuration formed by agents in C_1 . Since the clustering is nontrivial, the number of agents in C_1 is less than k . Then by induction, $|\hat{f}(\hat{p})| \geq \hat{\mu}(k - 1, d) > 0$. So there is at least one agent \vec{x}_i in \hat{p} such that $|\hat{f}_i(\hat{p})| \geq \frac{1}{\sqrt{k-1}}\hat{\mu}(k - 1, d)$.

Now take into account interactions between agents in adjacent clusters. By the condition of

fading attraction, there exists $d' > 0$ such that for any $ij \in E$ and any $d > d'$, we have

$$|\bar{g}_{ij}(d)| < \frac{1}{2k^{3/2}}\hat{\mu}(k-1, d) \quad (1.54)$$

Increase l , if necessary, so that the resulting distance between agents in any two adjacent clusters is greater than d' . So then $|f_i(p)| > \frac{1}{2\sqrt{k}}\hat{\mu}(k-1, d)$ and hence,

$$|f(p)| > \frac{1}{2\sqrt{k}}\hat{\mu}(k-1, d) > 0 \quad (1.55)$$

for any $p \in X(d) - X^b(d)$. Consequently $\mu_3(k, d) \geq \frac{1}{2\sqrt{k}}\hat{\mu}(k-1, d) > 0$. \square

We now prove theorem 1.4.3.

proof of theorem 1.4.3. If d is absent in any equilibrium, then there is a closed neighborhood I of d such that any distance in I is absent in any equilibrium. In the proof of lemma 1.4.4, we have showed that there is a positive number a such that if we let

$$\mu'_{ij}(N, d) := \inf\{|f(p)| \mid p \in X_{ij}(d) \cap P_a\} \quad (1.56)$$

then $\mu'_{ij}(N, d) = \mu_{ij}(N, d)$, and this holds for all $d \in I$. So it suffices to prove that $\mu'_{ij}(N, d)$ is continuous at d .

Given $\epsilon > 0$, there exists $l > 0$ such that if $|d - d'| < l$, then for each p in $X_{ij}(d)$, there is p' in $X_{ij}(d')$ with $|p - p'| < \epsilon$. On the other hand, given $\delta > 0$, there is $\epsilon > 0$ such that if both p and p' are in P_a , and if $|p - p'| < \epsilon$, then $|f(p) - f(p')| < \delta$. This is because each function \bar{g}_{ij} is bounded when restricted on the interval $[a, \infty)$, so given a positive number δ , there is $\epsilon' > 0$ such that $|\bar{g}_{ij}(d' + \epsilon'') - \bar{g}_{ij}(d')| < \delta$ for any $d' \geq a$, any $\epsilon'' \in (0, \epsilon')$ and any $ij \in E$.

Pick d' in $(d - l, d + l)$, and without loss of generality, we assume $\mu'(N, d') \geq \mu'(N, d)$. Let $\{p_k\}_{k \in \mathbb{N}}$ be a sequence of configurations in $X_{ij}(d) \cap P_a$ with $\lim_{k \rightarrow \infty} |f(p_k)| = \mu'(N, d)$. For each p_k ,

we pick a configuration $p'_k \in X_{ij}(d') \cap P_a$ with $|p'_k - p_k| < \epsilon$. So then $|f(p'_k) - f(p_k)| < \delta$ and hence,

$$|\inf_{k \in \mathbb{N}} |f(p'_k)| - \mu'_{ij}(d, N)| < \delta \quad (1.57)$$

On the other hand,

$$\inf_{k \in \mathbb{N}} |f(p'_k)| \geq \mu'_{ij}(N, d') \geq \mu'_{ij}(N, d) \quad (1.58)$$

which implies that $|\mu'_{ij}(N, d') - \mu'_{ij}(N, d)| < \delta$. Since this holds for each d' in $(d - l, d + l)$, we then prove the continuity of $\mu'_{ij}(N, d)$ at d . \square

There is a similar version of theorem 1.4.3 for $Y(r)$, and we state it below without a proof.

Theorem 1.4.5. *Let $r > 0$ be a radius absent in any equilibrium, then $\nu(N, r) > 0$ and there exists a closed neighborhood J of r such that $\nu(N, r') > \frac{1}{2}\nu(N, r)$ for any $r' \in J$.*

In the rest of this chapter, we assume that the closed intervals I and J are chosen in a way so that they are symmetric around d and r , respectively.

1.4.3 A lower bound for escaping time

Our goal here is to establish theorem 1.4.6 where we set up a lower bound for the period that a gradient flow takes to escape out of $X_{ij}(I)$ from $X_{ij}(d)$. In particular, we do this by establishing a lower bound for the width of $X_{ij}(I)$ and an upper bound for the escaping velocity. These two bounds are established in lemma 1.4.7 and lemma 1.4.9, respectively.

Theorem 1.4.6. *There is a positive number v_p associate with each gradient flow $\varphi_{\geq 0}(p)$ such that $v_p \geq |f(\varphi_t(p))|$ for all $t \geq 0$. Let $I = [d - l_\mu, d + l_\mu]$ be a closed interval in \mathbb{R}^+ . If there is a moment $t > 0$ at which $d_{ij} = d$, then it takes at least $\tau_\mu = l_\mu / \sqrt{2}v_p$ units of time for a gradient flow to escape out of $X_{ij}(I)$ from $X_{ij}(d)$*

We only consider the case p is not an equilibrium because otherwise, the gradient flow $\varphi_{\geq 0}(p)$ stays at p for all time. The proof of theorem 1.4.6 will be given after lemma 1.4.7 and lemma 1.4.9.

Lemma 1.4.7. *Let d and d' be two positive numbers with $d > d'$, let $D(X_{ij}(d), X_{ij}(d'))$ be the distance between $X_{ij}(d)$ and $X_{ij}(d')$, i.e.,*

$$D(X_{ij}(d), X_{ij}(d')) := \inf\{|p - p'| \mid p \in X_{ij}(d), p' \in X_{ij}(d')\} \quad (1.59)$$

then $D(X_{ij}(d), X_{ij}(d')) = (d - d')/\sqrt{2}$.

Proof. It is clear that $D(X_{ij}(d), X_{ij}(d')) \geq (d - d')/\sqrt{2}$ because

$$D(X_{ij}(d), X_{ij}(d')) \geq \sqrt{|\vec{x}_i - \vec{x}'_i|^2 + |\vec{x}_j - \vec{x}'_j|^2} \geq (d - d')/\sqrt{2} \quad (1.60)$$

On the other hand, for almost all configurations p in $X_{ij}(d)$, there will be a configuration p' in $X_{ij}(d')$ with $|p - p'| = (d - d')/\sqrt{2}$. This is simply achieved by fixing all the agents in p but moving \vec{x}_i and \vec{x}_j to get $(d - d')$ closer to each other along the line determined by themselves, then we will get a p' in $X_{ij}(d')$ with $|p - p'| = (d - d')/\sqrt{2}$ as long as the new positions of \vec{x}_i and \vec{x}_j are not occupied by other agents. But this holds for almost all configurations in $X_{ij}(d)$. \square

There is a similar version for $Y(r)$.

Lemma 1.4.8. *Let r, r' be two radii with $r > r' > 0$, and let $D(Y(r), Y(r'))$ be the distance between $Y(r)$ and $Y(r')$, then $D(Y(r), Y(r')) > r - r'$.*

Proof. The centroid of a configuration p is at the origin. So if $p = (\vec{x}_1, \dots, \vec{x}_N)$ is in $Y(r)$ while $p' = (\vec{x}'_1, \dots, \vec{x}'_N)$ is in $Y(r')$, then there are at least two pairs of agents, say (\vec{x}_1, \vec{x}'_1) and (\vec{x}_2, \vec{x}'_2) , such that $\vec{x}_i \neq \vec{x}'_i$, $i = 1, 2$. We may also assume that \vec{x}_1 and \vec{x}'_1 are the outermost agents in p and p' , respectively. So then $|p - p'| \geq |\vec{x}_1 - \vec{x}'_1| + |\vec{x}_2 - \vec{x}'_2| > |\vec{x}_1 - \vec{x}'_1| \geq r - r'$. \square

Let $I = [d - l_\mu, d + l_\mu]$ be a closed neighborhood of d , then by lemma 1.4.7 we have $D(X_{ij}(d), X_{ij}(d \pm l_\mu)) = l_\mu/\sqrt{2}$. This then establishes a lower bound for the distance that a gradient flow has to travel to escape out of $X_{ij}(I)$ from $X_{ij}(d)$. We now show that for each gradient flow $\varphi_{\geq 0}(p)$, the velocity $|f(\varphi_t(p))|$ has a positive upper bound.

Lemma 1.4.9. *Consider the RMA system described by equation (1.1). For each initial condition p , we let*

$$v_p := \sup\{|f(\varphi_t(p))| \mid t \geq 0\} \quad (1.61)$$

then v_p exists.

Proof. By condition of *fading attraction*, the function $\bar{g}_{ij}(d)$ is bounded above. On the other hand, by theorem 1.2.1, there exists $a > 0$ such that $\varphi_{\geq 0}(p) \subset P_a$. So the magnitude of the interaction between any two agents along the gradient flow is bounded above, and so is $|f(\varphi_t(p))|$ for any $t \geq 0$. \square

Theorem 1.4.6 is then proved by combining lemma 1.4.7 and lemma 1.4.9.

1.4.4 Proofs of theorem 1.4.1 and theorem 1.4.2

Proof of theorem 1.4.1. By theorem 1.4.3, there is a closed interval I of d such that $f(p) \geq \frac{1}{2}\mu_{ij}(N, d)$ for any $d \in I$. We may assume that $I = [d - l_\mu, d + l_\mu]$. By theorem 1.4.6, it takes at least $\tau_\mu = l_\mu / \sqrt{2}v_p$ units of time for the gradient flow to leave the dissipation zone $X_{ij}(I)$ from $X_{ij}(d)$, so during the period $[t, t + \tau_\mu]$, the loss of potential is given by

$$\Psi_t(p) - \Psi_{t+\tau_\mu}(p) = \int_t^{t+\tau_\mu} |f(\varphi_s(p))|^2 ds \geq \frac{1}{4}\mu_{ij}^2(N, d)\tau_\mu \quad (1.62)$$

This then completes the proof. \square

The arguments in the proof of theorem 1.4.1 can be directly used to prove theorem 1.4.2, so we omit the proof of theorem 1.4.2 here.

We end this section with a little discussion on the dissipation zone. For each k -agent system, by theorem 1.2.2 there is a distance d_k and a radius r_k such that if $d \geq d_k$ (or $r \geq r_k$), then d as a distance (or r as a radius) is absent in any equilibrium. Let D and R be two positive numbers that

satisfy

$$D \geq \max\{d_k | 2 \leq k \leq N\} \quad (1.63)$$

$$R \geq \max\{r_k | 2 \leq k \leq N\} \quad (1.64)$$

Here we abuse the notation R as it is already defined in the statement of theorem 1.3.4, yet we may assume that R is sufficiently large so that it can be applied for both cases.

Let d and r be chosen from $[D, \infty)$ and $[R, \infty)$ respectively. For each k -agent system, there correspond $\mu_{ij}(k, d)$ and $\nu(k, r)$, and we here define

$$\hat{\mu}_{ij}(d) := \min\{\mu_{ij}(k, d) | 2 \leq k \leq N\} \quad (1.65)$$

$$\hat{\nu}(r) := \min\{\nu(k, r) | 2 \leq k \leq N\} \quad (1.66)$$

The definitions of D , R , $\hat{\mu}_{ij}(d)$ and $\hat{\nu}(r)$ are useful for studying semi-diverging gradient flows as we will define in the next section.

1.5 Asymptotic behavior of a semi-diverging gradient flow

1.5.1 Definitions and theorem 1.5.1

In this section, we will first define the notion of semi-diverging gradient flow, and investigate certain properties associated with it. Let (l, ϵ) be a fixed pair of positive parameters for clustering, and let R be defined by equation (1.64). Fix a nontrivial clustering $\sigma(l, \epsilon)$, we say a gradient flow $\varphi_{\geq 0}(p)$ is **semi-diverging** with respect to $\sigma(l, \epsilon)$ if for any $t > 0$, there is a moment $t_0 > t$ such that the configuration $\varphi_{t_0}(p)$ admits $\sigma(l, \epsilon)$, and the radius of each cluster in $\varphi_{t_0}(p)$ is bounded above by R . Our goal in this section is to develop theorem 1.5.1.

Theorem 1.5.1. *Let $\sigma(l, \epsilon)$ be a nontrivial clustering, and let $\varphi_{\geq 0}(p)$ be a semi-diverging gradient flow with respect to $\sigma(l, \epsilon)$. There is a threshold L for l and a threshold T for time such that if $\varphi_{t_0}(p)$*

admits $\sigma(l, \epsilon)$ with $t_0 \geq T$ and $l \geq L$, then there is a moment $t \geq t_0$ such that $\varphi_{t_0}(p)$ will reach a dissipation zone $X_{ij}(d)$ with $d \in [L - 2R, L + 2R]$ for some $ij \in E$.

The theorem will be established after a sequence of lemmas and theorems, and the proof will be given at the end of this section.

1.5.2 Metric property of clustering

Our goal here is establish theorem 1.5.2. This theorem describes a metric property of a semi-diverging gradient flow, in particular, it relates radii of clusters to adjacent-clusters distances in configurations along a semi-diverging gradient flow.

Theorem 1.5.2. *Let $\sigma(l, \epsilon)$ be a nontrivial clustering, and let $\varphi_{\geq 0}(p)$ be a semi-diverging gradient flow with respect to $\sigma(l, \epsilon)$. There is a threshold L_1 for l and a threshold T for time such that if $\varphi_{t_0}(p)$ admits $\sigma(l, \epsilon)$ with $l \geq L_1$ and $t_0 \geq T$, then each radius of cluster will remain less than R along the gradient flow $\varphi_{\geq t_0}(p)$ as long as each adjacent-cluster distance keeps greater than L_1 .*

The proof of theorem 1.5.2 will be given after lemma 1.5.3.

Lemma 1.5.3. *Let $\sigma(l, \epsilon)$ be a nontrivial clustering, and let $\varphi_{\geq 0}(p)$ be a semi-diverging gradient flow with respect to $\sigma(l, \epsilon)$. Suppose $\varphi_{t_0}(p)$ admits $\sigma(l, \epsilon)$, and suppose the maximum radius of cluster in $\varphi_{t_0}(p)$ is R . Let $\hat{\nu}(R)$ be defined by equation (1.66), then there is a threshold L_1 for l and a fixed period $\tau_\nu > 0$ such that if $l \geq L_1$, then $|f(\varphi_t(p))| > \hat{\nu}(R)/4$ for any $t \in [t_0, t_0 + \tau_\nu]$.*

Proof. We assume that the radius of C_1 in $\varphi_{t_0}(p)$ is R , and we denote by $\varphi_{t_0}(\hat{p})$ the sub-configuration formed by agents in C_1 , so then $|\hat{f}(\varphi_{t_0}(\hat{p}))| > \hat{\nu}(R)$. Let $J := [R - \delta, R + \delta]$ and we assume $\hat{\nu}(r) > \frac{1}{2}\hat{\nu}(R)$ for any $r \in J$.

Recall in lemma 1.4.8, we have proved that if $p_1 \in Y(R)$ and $p_2 \in Y(R \pm \delta)$, then $|p_1 - p_2| > \delta$, in the lemma we implicitly assumed that both p_1 and p_2 are centered at the origin. We now consider one of its variations. Suppose p_1 is of radius R centered at \vec{c}_1 while p_2 is of radius $R \pm \delta$ centered at \vec{c}_2 , then

$$|p_1 - p_2| > |\vec{c}_1 - \vec{c}_2| + \delta > \delta \quad (1.67)$$

Let v_p be defined by equation (1.61), then $v_p > 0$, and we let $\tau_\nu := \delta/v_p$. Inequality (1.67) then implies that the radius of C_1 will be within interval J during the period $[t_0, t_0 + \tau_\nu]$.

By condition of fading attraction, there is a distance $\tilde{d} > 0$ such that for any $d > \tilde{d}$ and any $ij \in E$, we have

$$|\bar{g}_{ij}(d)| < \frac{1}{4N^{3/2}}\hat{\nu}(R) \quad (1.68)$$

Let $L_1 := \tilde{d} + 2v_p\tau_\nu + 2R$. Then if each adjacent-cluster distance in $\varphi_{t_0}(p)$ is greater than L_1 , then the distance between any two agents of adjacent clusters will be greater than $\tilde{d} + 2v_p\tau_\nu$ at time t_0 . Since $|f_i(\varphi_t(p))| < |f(\varphi_t(p))| \leq v_p$ for all $i = 1, \dots, N$ and all $t > 0$, the distance between any two agents of adjacent clusters will be greater than \tilde{d} during the period $[t_0, t_0 + \tau_\nu]$.

Let $f(\varphi_t(\hat{p}))$ be the restriction of $f(\varphi_t(p))$ to $\varphi_t(\hat{p})$, it is a sum of two parts: the intra-cluster part $f_A(\varphi_t(\hat{p}))$ and the inter-cluster part $f_B(\varphi_t(\hat{p}))$, i.e.,

$$f(\varphi_t(\hat{p})) = f_A(\varphi_t(\hat{p})) + f_B(\varphi_t(\hat{p})) \quad (1.69)$$

The intra-cluster part $f_A(\varphi_t(\hat{p}))$ is contributed by agents inside C_1 , so by previous notation, it is just $\hat{f}(\varphi_t(\hat{p}))$. We have chosen τ_ν so that if $t \in [t_0, t_0 + \tau_\nu]$, then $|f_A(\varphi_t(\hat{p}))| > \hat{\nu}(R)/2$. The inter-cluster part $f_B(\varphi_t(\hat{p}))$ is contributed by agents outside C_1 . The threshold L_1 is chosen so that $|f_B(\varphi_t(\hat{p}))| < \hat{\nu}(R)/4$ for any $t \in [t_0, t_0 + \tau_\nu]$. So then, during the period $[t_0, t_0 + \tau_\nu]$, we have $|f(\varphi_t(p))| > |f(\varphi_t(\hat{p}))| \geq |f_A(\varphi_t(\hat{p}))| - |f_B(\varphi_t(\hat{p}))| > \hat{\nu}(R)/4$. \square

We now prove theorem 1.5.2.

Proof of theorem 1.5.2. The theorem is trivially true if the clustering $\sigma(l, \epsilon)$ is agent-wise, so we assume otherwise.

The potential function $\Psi(p)$ is bounded below as p varies over P because each integral $\int_1^d \bar{g}_{ij}(x)dx$ is bounded below as d varies over \mathbb{R}^+ . This in particular implies that

$$\lim_{t \rightarrow \infty} \int_t^\infty |f(\varphi_s(p))|^2 ds = 0 \quad (1.70)$$

Let $\hat{\nu}(R) > 0$ be defined by equation (1.65), and let $\tau_\nu > 0$ be the period defined in the statement of lemma 1.5.3, then there is a moment $T > 0$ such that for any $t \geq T$, we have

$$\int_t^\infty |f(\varphi_s(p))|^2 ds < \frac{1}{16} \hat{\nu}^2(R) \tau_\nu \quad (1.71)$$

We now assume $t_0 > T$, $l > L_1$, and we assume that all adjacent-cluster distances keep greater than L_1 along the gradient flow $\varphi_{\geq t_0}(p)$. So then, if there is a moment $t \geq t_0$ at which the maximum radius of cluster reaches R , then by lemma 1.5.3, the loss of the potential of the gradient flow over the period $[t, t + \tau_\nu]$ will exceed $\hat{\nu}^2(R) \tau_\nu / 16$ which is a contradiction. \square

1.5.3 Metric property of configuration

Our goal here is to establish theorem 1.5.4. This theorem describes a metric property of a semi-diverging gradient flow, in particular, it relates the size of a configuration along a semi-diverging gradient flow to both radii of clusters and adjacent-cluster distances.

Theorem 1.5.4. *Let $\sigma(l, \epsilon)$ be a nontrivial clustering, and let $\varphi_{\geq 0}(p)$ be a semi-diverging gradient flow with respect to $\sigma(l, \epsilon)$. There is a threshold L_2 for l such that if $\varphi_{t_0}(p)$ admits $\sigma(l, \epsilon)$ with $l \geq L_2$ and if along the gradient flow $\varphi_{\geq t_0}(p)$,*

1. *each adjacent-cluster distance remains greater than L_2 .*
2. *each radius of cluster remains less than R .*

then there exists a number $b > 0$ such that $\varphi_{\geq t_0}(p) \subset P^b$.

Proof. We may as well assume $t_0 = 0$. Choose L_2 large enough so that the interaction between any two agents of adjacent clusters is an attraction. Let $\vec{c}_i(t) \in \mathbb{R}^n$ be the center of cluster C_i at time t , and let

$$\hat{\vec{c}}_i(t) := \begin{cases} \vec{c}_i(t) / |\vec{c}_i(t)| & \vec{c}_i(t) \neq 0 \\ 0 & \vec{c}_i(t) = 0 \end{cases} \quad (1.72)$$

We assume there are M clusters, and each cluster C_i contains n_i agents. Let $S := \{1, \dots, M\}$, then for each $i \in S$, and each subset $S' \subset S$, we define

$$\pi(i, S', t) := \langle \hat{c}_i(t), \frac{\sum_{j \in S'} n_j \vec{c}_j(t)}{\sum_{j \in S'} n_j} \rangle \quad (1.73)$$

it is understood as the projection of the center of agents in $\bigcup_{j \in S'} C_j$ along direction $\hat{c}_i(t)$. For each $m = 1, \dots, M$, we then define

$$\pi_m(t) := \max\{\pi(i, S', t) | i \in S, |S'| = m\} \quad (1.74)$$

each $\pi_m(t)$, as a function of t , is continuous and piecewise differentiable. It suffices for us to show $\pi_1(t)$ is bounded above for all $t > 0$ because by definition, $\pi_1(t)$ is the radius of $\varphi_t(p)$. The proof is then done by contradiction, i.e, we assume for any distance d , there is a moment $t_1 > 0$ such that $\pi_1(t_1) = d$.

We may as well assume that $d > \pi_1(0)$ and $\pi_1(t) < d$ for $t < t_1$. Suppose C_1 is the outermost cluster at time t_1 , then at the same moment, there is at least one cluster, say C_2 , adjacent to C_1 such that

$$\langle \hat{c}_1(t_1), \vec{c}_2(t_1) \rangle > d - 2R \quad (1.75)$$

because otherwise, all clusters that are adjacent to C_1 will pull back C_1 along direction $\hat{c}_1(t_1)$. Then by time reversing, there is a number $\delta > 0$ such that $\vec{c}_1(t) > \vec{c}_1(t_1)$ for any $t \in (t_1 - \delta, t_1)$ which contradicts to our assumption on t_1 . Inequality (1.75) then implies that

$$\pi_2(t_1) \geq \pi(1, \{1, 2\}, t_1) \geq d - 2R \quad (1.76)$$

We may assume d is sufficiently large so that $d - 2R > \pi_2(0)$. Then during the period $[0, t_1]$, there exists a moment t_2 such that $\pi_2(t_2) = d - 2R$ and $\pi_2(t) < d - 2R$ for $t < t_2$. Without loss of generality, we assume

$$\pi_2(t_2) = \pi(1, \{1, 2\}, t_2) = d - 2R \quad (1.77)$$

and

$$\langle \hat{\vec{c}}_1(t_2), \vec{c}_2(t_2) \rangle \leq \langle \hat{\vec{c}}_1(t_2), \vec{c}_1(t_2) \rangle \leq d \quad (1.78)$$

so then

$$\langle \hat{\vec{c}}_1(t_2), \vec{c}_2(t_2) \rangle \geq d - 2NR \quad (1.79)$$

Now apply the same arguments, we conclude that at the moment t_2 , there is at least one cluster C_3 adjacent to either C_1 or C_2 , or both, such that

$$\langle \hat{\vec{c}}_1(t_2), \vec{c}_3(t_2) \rangle \geq d - 2(N+1)R \quad (1.80)$$

this then, in turn, implies that

$$\pi_3(t_2) \geq \pi(1, \{1, 2, 3\}, t_2) \geq d - 2(N+1)R \quad (1.81)$$

Repeat the process, we then find a sequence of decreasing moments $t_1 \geq \dots \geq t_{M-1} > 0$ such that at each time t_k , there is an index $i \in S$ and a subset $S' \subset S$ with $|S'| = k+1$ such that for any $j \in S'$

$$\langle \hat{\vec{c}}_i(t_k), \vec{c}_j(t_k) \rangle \geq d - 2 \sum_{l=0}^{k-1} N^l R > 0 \quad (1.82)$$

Now consider the configuration $\varphi_{t_{M-1}}(p)$, inequality (1.82) then implies that the center of each cluster is on one side of the hyperplane that is perpendicular to $\hat{\vec{c}}_i(t_{M-1})$. But this is a contradiction because $\varphi_{t_{M-1}}(p)$ is centered at the origin. \square

1.5.4 Proof of theorem 1.5.1

Proof of theorem 1.5.1. Let D be defined by equation (1.63), so then $X_{ij}(D)$ is dissipation zone. Let L_1 , T be defined in the statement of theorem 1.5.2, let L_2 be defined in the statement of theorem 1.5.4, and we define $L := \max\{L_1, L_2, D + 2R\}$.

The proof is done by contradiction. i.e, we assume $\varphi_{t_0}(p)$ admits $\sigma(l, \epsilon)$ with $l \geq L$ and $t_0 \geq T$,

yet the flow $\varphi_{\geq t_0}(p)$ doesn't intersect $X_{ij}(I)$ for any $ij \in E$. By theorem 1.5.2, if each adjacent-cluster distance keeps greater than L_1 , then the radius of each cluster will be bounded above by R along the gradient flow $\varphi_{\geq t}(p)$. Then by theorem 1.5.4, the gradient flow $\varphi_{\geq t_0}(p)$ is contained in P^b for some $b > 0$.

Such a gradient flow will converge to the set of equilibria as we will see in the next section. However, the diameter of each equilibrium is bounded above by D , so there must exist a moment $t > t_0$ at which the gradient flow intersects $X_{ij}(D)$ for some $ij \in E$. Since $L \geq D + 2R$, so during the period $[t_0, t]$, the gradient flow intersects a dissipation zone $X_{ij}(d)$ for some $d \in [L - 2R, L + 2R]$ which is a contradiction. \square

1.6 Convergence of the gradient flow

Our goal in this section is to prove that each gradient flow associated with the model described by equation (1.1) converges to the set of equilibria. This will be done after theorem 1.6.1.

Theorem 1.6.1 (Swarm aggregation). *Consider the RMA system described by equation (1.1), for each configuration p , there exists a positive number $b > 0$ such that $\varphi_{\geq 0}(p)$ is contained in P^b .*

We will first assume theorem 1.6.1 to establish the convergence of the gradient flow, and then give a proof of it. Recall the ω -limit set of a configuration p is defined to be

$$\omega(p) := \bigcap_{t>0} \overline{\bigcup_{s \geq t} \varphi_s(p)} \quad (1.83)$$

By theorem 1.2.1 and theorem 1.6.1, for each initial condition p , there are two positive numbers a and b associated with p such that the gradient flow $\varphi_{\geq 0}(p)$ is contained in the compact set P_a^b .

We now show that the ω -limit set is nonempty and consists only of equilibria. Let $\{t_i\}_{i \in \mathbb{N}}$ be a sequence approaching to infinity, since $\{\varphi_{t_i}(p)\}_{i \in \mathbb{N}}$ is contained in a compact set P_a^b , there must exist at least one accumulation point. On the other hand, all accumulation points must be equilibria. Because if not, say there exists a configuration p' with $f(p') \neq 0$ and p' is the limit of the sequence

$\{\varphi_{t_i}(p)\}_{i \in \mathbb{N}}$. By continuity of the gradient field, there exists an open neighborhood U of p' and a time period τ such that $|f(\varphi_s(p''))| > \frac{1}{2}|f(p')|$ for any $p'' \in U$ and any $s \in [0, \tau]$. By passing to a subsequence, if necessary, we assume that the sequence $\{\varphi_{t_i}(p)\}_{i \in \mathbb{N}}$ is contained in U and $t_{i+1} - t_i > \tau$. So then

$$\int_0^\infty |f(\varphi_s(p))|^2 ds > \sum_{i \geq 1} \frac{1}{4} |f(p)|^2 \tau = \infty \quad (1.84)$$

which contradicts the fact that the potential function $\Psi(p)$ is bounded below as p varies over P . So at this moment, we have actually proved theorem 1.1, the main theorem of this chapter. The remaining section is devoted to the proof of theorem 1.6.1

Proof of theorem 1.6.1. The proof is done by contradiction, i.e, we assume that there exists a gradient flow $\varphi_{\geq 0}(p)$ such that the radius of the configuration $\varphi_t(p)$ can't be bounded above as t approaches to infinity. Then by theorem 1.3.4, there is an infinite sequence $\{\varphi_{t_i}(p), \sigma(l_i, \epsilon)\}_{t_i \rightarrow \infty}$ such that

1. each clustering $\sigma(l_i, \epsilon)$ on $\varphi_{t_i}(p)$ induces the same partition on V , the set of vertices in Γ .
2. the sequence $\{l_i\}_{i \in \mathbb{N}}$ approaches to infinity.
3. the radius of each cluster in $\varphi_{t_i}(p)$ is bounded above by R for all t_i .

Let L and T be defined in the statement of theorem 1.5.1. Let $I := [L - 2R, L + 2R]$, and we can assume that each distance $d \in I$ is absent in any equilibrium by increasing L . There is an index k_0 such that if $k \geq k_0$, then $t_k \geq T$ and $l_k \geq L$, we may assume that $k_0 = 1$ by passing to a subsequence. Then by theorem 1.5.1, for each t_k , there is a moment $t'_k > t_k$ such that the gradient flow falls into $X_{ij}(I)$ for some $ij \in E$ at time t'_k . For each $d \in I$, we have $\hat{\mu}_{ij}(d) > 0$, and I is a closed interval, so

$$\hat{\mu}_{ij}(I) := \min\{\hat{\mu}_{ij}(d) | d \in I\} > 0 \quad (1.85)$$

Let \tilde{I} be a thickened version of I defined by

$$\tilde{I} := [L - 2R - l_\mu, L + 2R + l_\mu] \quad (1.86)$$

with $l_\mu > 0$ chosen so that $\hat{\mu}_{ij}(\tilde{I}) \geq \hat{\mu}_{ij}(I)/2$. Let v_p be defined by equation (1.61), and let $\tau_\mu := l_\mu/\sqrt{2}v_p$. Then the loss of potential of the gradient flow exceeds $\hat{\mu}_{ij}^2(I)\tau_\mu/4$ over the period $[t'_k, t'_k + \tau_\mu]$. By passing to a subsequence, if necessary, we may assume that the two time sequences $\{t_k\}_{k \in \mathbb{N}}$ and $\{t'_k\}_{k \in \mathbb{N}}$ are interlacing, i.e.,

$$t_k < t'_k < t_{k+1} < t'_{k+1}, \quad \forall k \in \mathbb{N} \quad (1.87)$$

Let

$$\Delta := \min\{\hat{\mu}_{ij}^2(I)\tau_\mu/4 \mid ij \in E\} \quad (1.88)$$

then $\Delta > 0$, and the loss of potential along the gradient flow $\varphi_{\geq 0}(p)$ is bounded below by

$$\int_0^\infty |f(\varphi_t(p))|^2 dt > \sum_{k \in \mathbb{N}} \Delta = \infty \quad (1.89)$$

This contradicts to the fact that the potential function $\Psi(p)$ is bounded below over the configuration space. \square

In this chapter, we have showed that RMA systems with connected network topology are well-defined and have well behaviors in the sense that there is no collision of pairs of adjacent agents, and there is no escape of agent to infinity. If the network topology Γ is not connected, then we may divide the graph into connected components, and consequently decompose the multi-agent system into isolated sub-systems.

We also note that it is important for us to have a compact set of equilibria. It is well known that on a compact manifold, if each equilibrium is hyperbolic (i.e, we linearize the gradient flow at each equilibrium as $\dot{\vec{x}} = A\vec{x}$, then the matrix A is non-singular, there is an equivariant version as we will introduce in the next chapter), then there are only finitely many equilibria. Since our configuration space is non-compact, so the fact that the set of equilibria is a compact global attractor is an indispensable condition for us to have this finiteness property. Many problems and applications, including counting number of critical formations, computing Euler characteristic of the configuration space, and etc. depends on the result we developed in this chapter.

Chapter 2

Index/Co-index Formula

2.1 Definitions and main theorem

In this chapter, we consider a special class of reciprocal multi-agent (RMA) systems by equipping it with a particular type of Laman graph. It is well known that the family of Laman graphs describes the class of minimally rigid systems of vertices and edges in the plane. A formal definition of a Laman graph can be described by a combinatorial condition.

Laman graph. An undirected graph Γ on N vertices is said to be a Laman graph if each k -vertex subgraph of Γ , $1 \leq k \leq N$, has at most $(2k - 3)$ edges, while the whole graph has exactly $(2N - 3)$ edges.

A Laman graph Γ is **rigid**, i.e, if we place the vertices of Γ on a plane in a general position (i.e, coordinates of vertices are algebraically independent over \mathbb{R}), then only rotations and translations will preserve the lengths of all graph edges. A Laman graph Γ is also **minimally rigid**, i.e, if we remove an edge from the graph, then the resulting graph won't be rigid anymore.

Because of this property of minimal rigidity, network topologies in multi-agent systems are often designed to be Laman graphs. In many cases, they are most effective and parsimonious way for agents to communicate with each other to maintain rigid formation in the plane. Also in problems concerning operations on network topology such as merging and splitting, minimal rigid graphs are more convenient to deal with. We here refer readers to [16] for works about maintaining minimal rigidity in multi-agent systems with time varying network topologies.

In this chapter, we consider a special class of Laman graphs, as we call Laman graphs of type-I (LGT-I). We show that if a RMA system is equipped with a LGT-I, then there is a geometric characterization of each equilibrium associated with the RMA system. This characterization decomposes an equilibrium into union of sub-configurations, each of which is a critical line formation. Moreover, there is an index/co-index formula, as we will establish in this chapter, associated with this geometric characterization. The formula says that the Morse index/co-index of an equilibrium can be computed by summing up Morse indices/co-indices of the associated sub-configurations. In this section, we will make key definitions and state the main theorem of this chapter. We start with the definition of a LGT-I.

Laman graph of Type-I (LGT-I). A LGT-I can be defined inductively via a special Henneberg construction of type-I. Start with an edge, we then join a new vertex, at each step, to two adjacent existing vertices via two new edges. We remind the readers that two vertices are said to be **adjacent** if there is an edge in between. A LGT-I has multiple ways of Henneberg construction, and in fact, it is “base-edge-irrelevant” in the sense that we can choose an arbitrary edge in a LGT-I as a base edge to start with. We will prove this property in section 2.2.

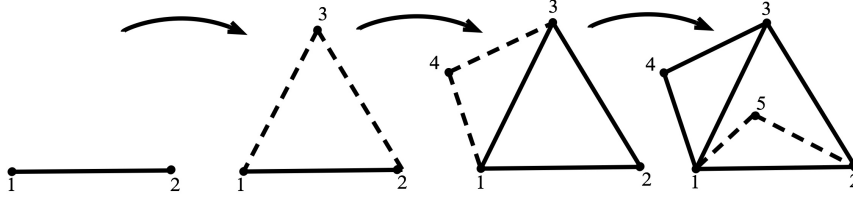


Figure 8: An example of a LGT-I with a particular Henneberg construction. Start with the base edge 12, we then subsequently join vertices 3, 4 and 5 by choosing two existing, adjacent vertices.

Let $\Gamma = (V, E)$ be a LGT-I on N vertices, and let $V(i) = \{j \in V | ij \in E\}$ be the set of vertices adjacent to vertex i . We recall the equations of motion, for agents $\vec{x}_1, \dots, \vec{x}_N \in \mathbb{R}^n$ with $N > n$, take the form

$$\ddot{\vec{x}}_i = \sum_{j \in V(i)} g_{ij}(d_{ij})(\vec{x}_j - \vec{x}_i), \quad i = 1, \dots, N \quad (2.1)$$

Each $g_{ij} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous differentiable function, and we require $g_{ij} = g_{ji}$ for all $ij \in E$. We still assume that each g_{ij} , $ij \in E$, satisfies conditions of strong repulsion and fading attraction as we defined in chapter I.

Strong repulsion. $\lim_{d \rightarrow 0} dg_{ij}(d) = -\infty$ and $\lim_{d \rightarrow 0} \int_d^1 xg_{ij}(x)dx = -\infty$.

Fading attraction. $g_{ij}(d) > 0$ if $d \gg 1$ and $\lim_{d \rightarrow \infty} dg_{ij}(d) = 0$.

In this chapter, for our convenience of dealing with sub-configurations, we do not restrict ourselves to configurations with zero centroid. So the configuration space is defined by

$$P := \{(\vec{x}_1; \dots; \vec{x}_N) \in \mathbb{R}^{n \times N} | \vec{x}_i \neq \vec{x}_j, \forall ij \in E\} \quad (2.2)$$

The potential function Ψ over P associated with system (2.1) is still given by

$$\Psi(\vec{x}_1, \dots, \vec{x}_N) := \sum_{ij \in E} \int_1^{d_{ij}} x g_{ij}(x) dx \quad (2.3)$$

We observe that Ψ depends only on relative distances of agents, so it is invariant if we rotate or translate a configuration. Naturally there is a group action of special Euclidean group acting on a configuration, emphasizing the invariance of rigid motion as we define below

Group action of rigid motion. Let $SE(n)$ be the special Euclidean group, an element $\alpha \in SE(n)$ can be uniquely expressed as the composition of a rotation and a translation, i.e,

$$\alpha = t_{\vec{v}} \cdot \theta \quad (2.4)$$

where θ is in the special orthogonal group $SO(n)$ and $t_{\vec{v}}$ is the translation along a vector $\vec{v} \in \mathbb{R}^n$. We now define the $SE(n)$ -action on P by sending $\alpha \in SE(n)$ and $p = (\vec{x}_1; \dots; \vec{x}_N) \in P$ to

$$\alpha \cdot p := (\theta \vec{x}_1 + \vec{v}; \dots; \theta \vec{x}_N + \vec{v}) \quad (2.5)$$

Let \mathcal{O}_p be the orbit of p with respect to the $SE(n)$ -action, then we have

$$\Psi(p) = \Psi(\alpha \cdot p) \quad (2.6)$$

for any $\alpha \in SE(n)$.

In this chapter, we will often deal with subgraphs and sub-configurations, so we here define some useful notations. Let $\Gamma_i = (V_i, E_i)$ be a subgraph of Γ . Suppose $V_i := \{i_1, \dots, i_k\}$, we then let

$$P_i := \{(\vec{x}_{i_1}, \dots, \vec{x}_{i_k}) | \vec{x}_{i_j} \neq \vec{x}_{i_k}, \forall i_j i_k \notin E_i\} \quad (2.7)$$

be the space of sub-configurations formed by agents \vec{x}_{i_j} with $i_j \in V_i$. Let Ψ_i be the **sub-potential**

of Ψ with respect to Γ_i defined over P_i . To be explicit, let p_i be a sub-configuration in P_i , then

$$\Psi_i(p_i) := \sum_{i_j i_k \in E_i} \int_1^{d_{i_j i_k}} x g_{i_j i_k}(x) dx \quad (2.8)$$

In other words, if the agents $\vec{x}_{i_1}, \dots, \vec{x}_{i_k}$ form an isolated sub-system with Γ_i its network topology, then Ψ_i will be the associated potential function. Similarly, we define the $SE(n)$ -action on a sub-configuration p_i by sending $\alpha = t_{\vec{v}} \cdot \theta$ and $p_i = (\vec{x}_{i_1}, \dots, \vec{x}_{i_k})$ to

$$\alpha \cdot p_i := (\theta \vec{x}_{i_1} + \vec{v}; \dots; \theta \vec{x}_{i_k} + \vec{v}) \quad (2.9)$$

and we denote by \mathcal{O}_{p_i} the orbit of p_i with respect to the $SE(n)$ -action.

Let $f(p)$ be the gradient vector field associated with potential function $\Psi(p)$, i.e.,

$$f(p) := -\nabla \Psi(p) \quad (2.10)$$

As a consequence of $\Psi(p) = \Psi(\alpha \cdot p)$, we have

$$\alpha \cdot f(p) = f(\alpha \cdot p) \quad (2.11)$$

for each $\alpha \in SE(n)$. This, in particular, implies that if p is an equilibrium associated with Ψ , then so is any p' in \mathcal{O}_p . In other words, it is inevitable that we have continuum equilibria. We will now introduce the notion of hyperbolicity of a critical submanifold and the notion of Morse index/co-index of a critical submanifold.

Hyperbolic critical submanifold. Let $\Psi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a C^2 -function. A point $p \in \mathbb{R}^m$ is said to be a critical point, or an equilibrium, if the derivative $d\Psi$ vanishes at p . Suppose K is a connected submanifold in \mathbb{R}^m consisting exclusively of critical points. We say K is hyperbolic if the Hessian of Ψ , denoted by $\mathcal{H}(\Psi)$, is nondegenerate when restricted to the normal space $N_p K$ for each $p \in K$. If there are only finitely many connected critical manifolds, and if each of them is hyperbolic, then

the function Ψ is said to be a **Morse-Bott function**. If, in addition, the potential function Ψ is an equivariant function with respect to a connected Lie group, then each connected critical manifold will be an orbit with respect to the group action. In this case, the function Ψ is said to be an **equivariant Morse function**, and each hyperbolic critical manifold is said to be a hyperbolic **critical orbit**.

Morse index/co-index of a hyperbolic critical submanifold. Let $\Psi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a smooth function, and let K be a hyperbolic critical submanifold. The Morse index/co-index of K with respect to Ψ , denoted by $\iota_-(K)/\iota_+(K)$, is defined to be the number of negative/positive eigenvalues of $\mathcal{H}(\Psi)$ at some $p \in K$. (This definition is independent of the choice of p since K is connected and hyperbolic.) An equation that relates Morse index, Morse co-index and the dimension of K is given by $\iota_-(K) + \iota_+(K) + \dim K = m$.

We are now ready to state the main theorem of this chapter, this theorem gives a geometric characterization of each equilibrium associated with the potential function Ψ defined by equation (2.3). Moreover, this theorem provides a formula for computing the Morse index/co-index of a hyperbolic critical orbit.

Theorem 2.1. *This theorem consists of two parts:*

Part I. Let $\Gamma = (V, E)$ be a LGT-I, and let $p = (\vec{x}_1, \dots, \vec{x}_N)$ be an equilibrium associated with the potential function Ψ . Then for each p , there is a unique partition, as we call the canonical partition, of E into disjoint, nonempty subsets E_1, \dots, E_k of E , satisfying three conditions

- *Combinatorial condition. If we let $\Gamma_i = (V_i, E_i)$ be a subgraph of Γ by restricting Γ to the set of edges E_i , then the subgraph Γ_i is a LGT-I.*
- *Geometric condition. All agents \vec{x}_{i_j} with $i_j \in V_i$ are aligned. Let p_i be the sub-configuration formed by \vec{x}_{i_j} with $i_j \in V_i$, and let Ψ_i be the sub-potential of Ψ with respect to Γ_i , then p_i is an equilibrium associated with Ψ_i .*
- *Maximality in a lattice of partitions. If $E = \bigcup_{i=1}^{k'} E'_i$ is another partition that satisfies the two conditions above, then it is a refinement of the partition $E = \bigcup_{i=1}^k E_i$. In other words, each E'_i ,*

$i = 1, \dots, k'$, is a subset of E_j for some $j = 1, \dots, k$.

Part II. Let p be an equilibrium associated with Ψ . Let p_i and Ψ_i , $i = 1, \dots, k$, be sub-configurations and sub-potentials, respectively, with respect to the canonical partition of E . The critical orbit \mathcal{O}_p is hyperbolic with respect to Ψ if and only if each critical orbit \mathcal{O}_{p_i} is hyperbolic with respect to Ψ_i . Let $\iota_-(\mathcal{O}_{p_i})/\iota_+(\mathcal{O}_{p_i})$ be the Morse index/co-index of \mathcal{O}_{p_i} with respect to Ψ_i , then the Morse index/co-index of \mathcal{O}_p with respect to Ψ is given by the **index/co-index formula**:

$$\begin{aligned}\iota_-(\mathcal{O}_p) &= \sum_{i=1}^k \iota_-(\mathcal{O}_{p_i}) \\ \iota_+(\mathcal{O}_p) &= \sum_{i=1}^k \iota_+(\mathcal{O}_{p_i})\end{aligned}\tag{2.12}$$

In other words, the index/co-index formula is compatible with the canonical partition of E .

The rest of this chapter is organized as follows. In section 2.2, we will prove part I of theorem 2.1. We show that there is a unique partition, as we call the canonical partition of E associated with a configuration p , that satisfies the three conditions defined in part I of theorem 2.1. In section 2.3, we will investigate an important geometric property associated with the canonical partition. As the partition gives rise to a family of sub-configurations, we exploit the following question: whether we can perturb one sub-configuration while finding displacements for other sub-configurations so that their shapes are preserved? The answer is yes, and the proof will be given in section 2.3. In section 2.4, we will compute the Hessian matrix at an equilibrium, and establish sufficient and necessary conditions for a critical orbit to be hyperbolic. In particular, we relate the hyperbolicity of a critical orbit to hyperbolicities of critical orbits of sub-configurations. In section 2.5, we will use the results we developed in previous sections to establish the index/co-index formula. An important application of the index/co-index formula on strongly nondegenerate configurations will also be investigated.

2.2 The canonical partition of edges associated with an equilibrium

Our goal in this section is to establish the first part of theorem 2.1. Our plan is to first construct a partition of E associated with a configuration p , and then verifies the three conditions in the first part of theorem. So this section is divided into three parts: in section 2.1, we define the canonical partition of E associated with p in a way that it automatically satisfies the combinatorial condition, then in section 2.2, we will show that the canonical partition satisfies the geometric condition. In section 2.3, we will investigate a lattice of partitions, and show that the canonical partition is greatest element in this lattice.

2.2.1 The canonical partition associated with an arbitrary configuration

Let $\Gamma = (V, E)$ be a LGT-I, and let p be a configuration. Choose a Henneberg construction of Γ , and label the vertices with respect to the order of the construction. Let $\Gamma^k = (V^k, E^k)$ be the restriction of Γ to vertices $V^k := \{1, \dots, k\}$. We now define the **canonical partition of E associated with p** by induction on k .

Base case. Start with Γ^1 , since $E^1 = \{12\}$ is a singleton, there is only the trivial partition of E^1 .

Inductive step. Suppose we have decomposed E^{k-1} , $k > 1$, into union of disjoint, nonempty subsets E'_1, \dots, E'_m , and we now define the partition of E^k . Let i and j be two vertices in V^{k-1} that the vertex k joins to, and we may assume that the edge ij is contained in subset E'_1 . There are two cases about how we partition E^k

Case I. If \vec{x}_i , \vec{x}_j and \vec{x}_k are aligned, then we update E'_1 by adding ik and jk into it. So then after the update, we have $E^k = E'_1 \cup \dots \cup E'_m$.

Case II. If \vec{x}_i , \vec{x}_j and \vec{x}_k are not aligned, then the partition of E^k is given by $E^k = E'_1 \cup \dots \cup E'_m \cup \{ik\} \cup \{jk\}$.

The induction then defines the canonical partition of E associated with p . This definition

apparently depends on the choice of a Henneberg construction of Γ , yet it doesn't.

Lemma 2.2.1. *The canonical partition of E associated with p is independent of the choice of a Henneberg construction of graph Γ .*

Proof. The proof is done by induction on the number agents.

Base case. The lemma is trivially true in the case $N = 2$ because there is only one Henneberg construction.

Inductive step. Assume the lemma hold for $N < m - 1$, we prove for the case $N = m$. We first choose a Henneberg construction, call it τ_1 . Label the vertices of the graph with respect to the order of τ_1 . Let i and j be the two vertices that the last vertex m joins to during τ_1 .

Now choose another Henneberg construction, call it τ_2 . Notice that the vertex m has to join to vertices i and j via edges im and jm . This is because vertex i and vertex j are the only two vertices that are adjacent to m , it then also implies that there is no vertex that will joint to m during the Henneberg construction τ_2 . In other words, it doesn't matter when we join m to the graph as long as the edge ij is formed during τ_2 . More importantly, the change in the order of the construction won't affect the canonical partition of E given by τ_2 . So we may as well assume that the vertex m is the last vertex added into Γ during τ_2 .

Let $\Gamma' = (V', E')$ be a subgraph of Γ defined by restricting Γ on vertices $V' = \{1, \dots, m - 1\}$. Since Γ' is a LGT-I, there are two Henneberg constructions τ'_1 and τ'_2 of Γ' by simply restricting τ_1 and τ_2 to Γ' .

By induction, the two Henneberg constructions τ'_1 and τ'_2 of Γ' give rise to the same canonical partition of E' as $E' = \bigcup_{i=1}^{k'} E'_i$. On the other hand, both τ_1 and τ_2 agree on how the two edges im and jm update the canonical partition of E' , so then τ_1 and τ_2 coincide with each other. \square

Let E_1, \dots, E_k be the family of disjoint, nonempty subsets that corresponds to the canonical partition of E associated with p . Let V_i be the set of vertices associated with E_i . By construction of the canonical partition, each subgraph $\Gamma_i := (V_i, E_i)$ is a LGT-I. So the canonical partition of E associate with p satisfies the combinatorial condition in part I of theorem 2.1. We end this section

with a simple example illustrating the canonical partition associated with p .

Example. Let $\Gamma = (V, E)$ be a graph with

$$\begin{aligned} V &:= \{1, 2, 3, 4, 5, 6\} \\ E &:= \{12, 13, 14, 16, 23, 34, 35, 45, 46\} \end{aligned} \tag{2.13}$$

A Henneberg construction of Γ is given by

	vertices		edges
Start with	1, 2	and	12
Join	3	via	13, 23
Join	4	via	14, 34
Join	5	via	35, 45
Join	6	via	16, 46

The order the vertices of Γ is with respect to this Henneberg construction. Now consider a configuration p illustrated in figure 9. We now use the Henneberg construction to determine the canonical partition of E associated with p .

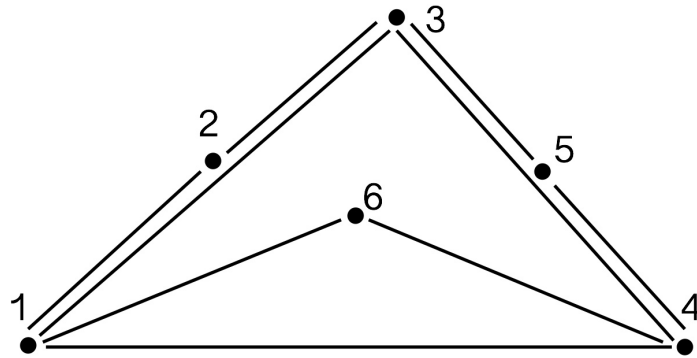


Figure 9: A configuration p with agents \vec{x}_1 , \vec{x}_2 and \vec{x}_3 aligned, and \vec{x}_3 , \vec{x}_4 and \vec{x}_5 aligned.

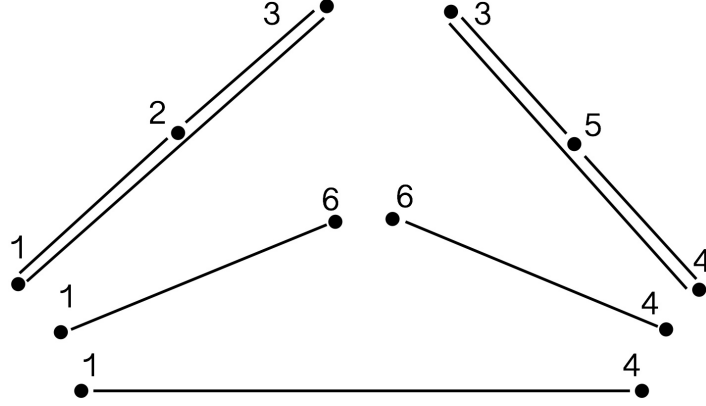


Figure 10: The canonical partition of E associated with p

	joining vertices	partition of edges	
Base case	1, 2	$\{12\}$	
Inductive step	3	$\{12, 13, 23\}$	
	4	$\{12, 13, 23\} \cup \{14\} \cup \{34\}$	(2.14)
	5	$\{12, 13, 23\} \cup \{34, 35, 45\} \cup \{14\}$	
	6	$\{12, 13, 23\} \cup \{34, 35, 45\} \cup \{14\} \cup \{16\} \cup \{46\}$	

2.2.2 The canonical partition associated with an equilibrium

In this part, we assume p is an equilibrium associated with the potential function Ψ , and we show that the canonical partition of E associate with p satisfies the geometric condition defined in part I of theorem 2.1.

Theorem 2.2.2. *Let Γ be a LGT-I, and let g_{ij} , $ij \in E$, be a family of interaction laws that satisfy strong repulsion and fading attraction. Let p be an equilibrium associated with Ψ . Let p_i and Ψ_i , $i = 1, \dots, k$, be sub-configurations and sub-potentials, respectively, with respect to the canonical partition of E . Then each p_i is an equilibrium associated with Ψ_i .*

Proof. The proof goes along with a Henneberg construction and will be done by induction on the number of agents.

Base case. If p is an equilibrium of two agents \vec{x}_1 and \vec{x}_2 , then $g_{12}(d_{12})$ has to vanish.

Inductive step. Assume the lemma holds for any N with $N < m$, and we prove for the case $N = m$. We assume that the vertices of Γ are labeled with respect to the order of a chosen Henneberg construction. So, in particular, vertex m is the last vertex we add into the graph. Let i and j be the two vertices that m joins to. There are two different cases depending on whether or not the two vectors $\vec{x}_m - \vec{x}_i$ and $\vec{x}_m - \vec{x}_j$ are linearly independent.

Case I. If the two vectors $\vec{x}_m - \vec{x}_i$ and $\vec{x}_m - \vec{x}_j$ are linearly independent, then $g_{im}(d_{im})$ and $g_{jm}(d_{jm})$ have to vanish because

$$\dot{\vec{x}}_m = g_{im}(d_{im})(\vec{x}_i - \vec{x}_m) + g_{jm}(d_{jm})(\vec{x}_j - \vec{x}_m) = 0 \quad (2.15)$$

Let $\Gamma' := (V', E')$ be a subgraph of Γ defined by restricting Γ to vertices $V' := \{1, \dots, m-1\}$. Let p' be the sub-configuration formed by agents $\vec{x}_1, \dots, \vec{x}_{m-1}$ and let Ψ' be the sub-potential of Ψ with respect to Γ' .

Since agent \vec{x}_m doesn't interact agents in p' at all, the sub-configuration p' is an equilibrium associated with the potential Ψ' . Let E'_1, \dots, E'_k be the family of disjoint subsets of E' that correspond to the canonical partition of E' associated with p' . This family of subsets, together with $\{im\}$ and $\{jm\}$, forms the canonical partition of E associated with p .

Let V'_i be the set of vertices associated with E'_i . By induction, the sub-configuration p'_i formed by agents \vec{x}_{i_j} , $i_j \in V'_i$, is an equilibrium associated with Ψ_i where Ψ_i is the sub-potential of Ψ with respect to $\Gamma'_i := (V'_i, E'_i)$. On the other hand, the two-agent system formed by $\{\vec{x}_i, \vec{x}_m\}$, or by $\{\vec{x}_j, \vec{x}_m\}$, reaches its own equilibrium because $g_{im}(d_{im}) = g_{jm}(d_{jm}) = 0$. This then establishes the result for the first case.

Case II. Now assume the two vectors $\vec{x}_m - \vec{x}_i$ and $\vec{x}_m - \vec{x}_j$ are linearly dependent. Without loss of generality, we assume the three edges ij , im and jm are contained in the subset $E_1 \subset E$. Since the

agent \vec{x}_m is balanced, i.e.,

$$g_{im}(d_{im})(\vec{x}_i - \vec{x}_m) + g_{jm}(d_{jm})(\vec{x}_j - \vec{x}_m) = 0 \quad (2.16)$$

So the interaction between \vec{x}_i and \vec{x}_m and the interaction between \vec{x}_j and \vec{x}_m have the same magnitude, but with opposite directions.

We now remove the agent \vec{x}_m and hence, annihilate the interaction between \vec{x}_i and \vec{x}_m and the interaction between \vec{x}_j and \vec{x}_m . Instead, we modify the interaction between \vec{x}_i and \vec{x}_j . Let

$$a_m := g_{im}(d_{im})d_{im} = g_{jm}(d_{jm})d_{jm} \quad (2.17)$$

and we assume that the modified interaction function \tilde{g}_{ij} between \vec{x}_i and \vec{x}_j takes the value $g_{ij}(d_{ij}) + a_m/d_{ij}$ at distance d_{ij} . Then all the remaining agents $\vec{x}_1, \dots, \vec{x}_{m-1}$ will still be balanced.

In other words, if we still let $\Gamma' = (V', E')$ be the subgraph defined by restricting Γ to vertices $V' = \{1, \dots, m-1\}$, and let p' be the sub-configuration formed by agents $\vec{x}_1, \dots, \vec{x}_{m-1}$, then p' is an equilibrium associated with the potential Ψ' where Ψ' is the sub-potential of Ψ with respect to Γ' , but with g_{ij} substituted by \tilde{g}_{ij} .

Let $\Gamma'_l := (V'_l, E'_l)$, $l = 1, \dots, k'$, be subgraphs with respect to the canonical partition of E' associated with p' . By induction, each sub-configuration p'_l formed by the agents \vec{x}_{l_j} , $l_j \in V'_l$, is an equilibrium associated with the potential Ψ'_l where Ψ'_l is the sub-potential of Ψ' with respect to Γ'_l . By comparing, the canonical partition of E with the canonical partition of E' , we have $k' = k$ and by relabeling $E'_1, \dots, E'_{k'}$ if necessary, we have

$$\begin{aligned} E_1 &= E'_1 \cup \{im\} \cup \{jm\} \\ E_l &= E'_l, \quad \forall l = 2, \dots, k \end{aligned} \quad (2.18)$$

Let Ψ_l be the sub-potential of Ψ with respect to the subgraph $\Gamma_l = (V_l, E_l)$, then $\Psi_l = \Psi'_l$ for any $l = 2, \dots, k$. So if p'_l is an equilibrium associated with Ψ'_l , then it is an equilibrium associated with Ψ_l as well.

Let p_1 be the sub-configuration formed by agents $x_{1_j}, 1_j \in V_1$. If p'_1 is an equilibrium associated with Ψ'_1 , then p_1 will be an equilibrium associated with Ψ_1 . This holds because we can reverse the modification of $g_{ij}(d_{ij})$ and retrieve the agent \vec{x}_m , together with the interaction between \vec{x}_i and \vec{x}_m , and the interaction between \vec{x}_j and \vec{x}_m . This then establishes the result for the second case. \square

So up till now, we have showed that the canonical partition of E associated with an equilibrium p satisfies the first two conditions in part I of theorem 2.1.

2.2.3 Maximality of the canonical partition

In this part, we will show that the canonical partition of E satisfies the maximality condition defined in part I of theorem 2.1. We will first make an important observation about a LGT-I.

Lemma 2.2.3. *Let $\tilde{\Gamma}$ be a subgraph of Γ , and we assume both Γ and $\tilde{\Gamma}$ are LGT-I. Then there exists a Henneberg construction of Γ with $\tilde{\Gamma}$ its top priority, i.e, the subgraph $\tilde{\Gamma}$ is built-up prior to any other vertices and edges in Γ during the construction.*

Proof. The proof is done by induction on the number of agents.

Base case. In the case $N = 2$, the lemma is trivial true.

Inductive step. Assume the lemma holds for $N < m$, we prove for the case $N = m$. Choose a Henneberg construction, and label the vertices with respect to the order of the construction. Let i and j be the two vertices that the last vertex m joins to. We let Γ' be the subgraph defined by restricting Γ on vertices $\{1, \dots, m-1\}$. There are two cases depending whether or not the subgraph $\tilde{\Gamma}$ contains the vertex m .

Case I. If $\tilde{\Gamma}$ doesn't contain vertex m , by induction we can choose a Henneberg construction of Γ' with $\tilde{\Gamma}$ its top priority. Then we build up Γ by simply adding the vertex m to Γ' via the two edges im and jm .

Case II. If $\tilde{\Gamma}$ contains vertex m , then it also contains the two vertices i and j and the three edges ij , im and jm since $\tilde{\Gamma}$. If we remove vertex m , together with the two edges im and jm from $\tilde{\Gamma}$, then the remaining graph $\tilde{\Gamma}'$ is still a LGT-I, and is a subgraph of Γ' . So by induction, there is a Henneberg

construction of Γ' with $\tilde{\Gamma}'$ its top priority. We now modify this Henneberg construction to build up Γ . The modification goes as follows: right after setting up $\tilde{\Gamma}'$, we pause the construction and join the vertex m to the two vertices i and j via the two edges im and jm , then we resume the Henneberg construction. \square

Remark. Let Γ' be a subgraph consisting of two vertices with an edge in between, then lemma 2.2.3 establishes the “base-edge-irrelevant” property of a LGT-I as we advertised earlier in the definition of LGT-I.

We will now prove that the canonical partition of E satisfies the maximality condition.

Theorem 2.2.4. *Let p be a configuration, and let $\sigma = (E_1, \dots, E_k)$ be a partition of E associated with p , satisfying two conditions:*

- *each subgraph $\Gamma_i = (V_i, E_i)$ is a LGT-I where V_i is the set of vertices associated with E_i .*
- *all agents \vec{x}_{i_j} with $i_j \in V_i$ are aligned.*

Let Σ be the set of partitions of E associated with p that satisfy the two conditions above. Define a partial order on Σ as follows. Suppose $\sigma = (E_1, \dots, E_k)$ and $\sigma' = (E'_1, \dots, E'_{k'})$ are two different elements in Σ . We denote by $\sigma \succ \sigma'$ if σ' is a refinement of σ , i.e, $k < k'$ and for each $i = 1, \dots, k'$, there is a $j = 1, \dots, k$ such that $E'_i \subset E_j$. Then with respect to this partial order, the canonical partition of E is the greatest element in Σ .

Proof. Let $\sigma = (E_1, \dots, E_k)$ be the canonical partition of E , then $\sigma \in \Sigma$. Suppose $\sigma' = (E'_1, \dots, E'_{k'})$ is another element in Σ , we show each E'_i is contained in some E_j . Let V_i and V'_i be the set of vertices associated with E_i and E'_i , respectively. Since the subgraph $\Gamma'_i = (V'_i, E'_i)$ is a LGT-I, we choose a Henneberg construction of Γ'_i , and label the vertices of $V'_i = \{i_1, \dots, i_{k_i}\}$ with respect to the order of the construction.

Now suppose $i_1 i_2$, the base edge of Γ'_i , is contained in E_j , consequently i_1 and i_2 are in V_j . We now show by induction that i_3, \dots, i_{k_i} are all contained in V_j . Suppose at this moment, vertices $\{i_1, \dots, i_{m-1}\}$, $3 \leq m \leq k_i$, are contained in V_j , we show that the vertex i_m is also contained in V_j .

Let i_a and i_b be the two vertices that i_m joins to with respect to the Henneberg construction of Γ'_i . Since the three agents \vec{x}_{i_m} , \vec{x}_{i_a} and \vec{x}_{i_b} are aligned as they are contained in V'_i . Then by definition of the canonical partition of E , the two edges $i_m i_a$ and $i_m i_b$ are in E_j because E_j contains $i_a i_b$ by induction. So the vertex i_m is contained in V_j . \square

Remark. Suppose p' is any other configuration in the orbit \mathcal{O}_p , then the canonical partition of E associated with p' agrees with the canonical partition of E associated with p . So the canonical partition of E applies to the whole orbit.

At this moment, we have showed that the canonical partition of E associated with p satisfies all the three conditions in part I of theorem 2.1. The uniqueness directly follows the maximality.

2.3 Perturbing one sub-configuration while preserving shapes of others

2.3.1 Theorem 2.3.1

Let p be a planar configuration, let p_1, \dots, p_k be sub-configurations of p with respect to the canonical partition of E . In this section, we will exploit the following question: is it possible that we perturb one sub-configuration while preserving the shapes of others?

Theorem 2.3.1. *Let Γ be a LGT-I, and let p be a planar configuration. Let $\Gamma_i = (V_i, E_i)$ and p_i , $i = 1, \dots, k$, be subgraphs and sub-configurations with respect to the canonical partition of E associated with p . Then for each p_i , there is an open neighborhood $W_i \subset \mathbb{R}^{2 \times |V_i|}$ of p_i such that if p_i is perturbed within W_i , then there is a unique displacement $\delta \vec{x}_j \in \mathbb{R}^2$ for each agent \vec{x}_j with $j \notin V_i$, satisfying two conditions*

Shape preserving. If we update the position of \vec{x}_j to $\vec{x}_j + \delta \vec{x}_j$ for each $j \notin V_i$, the distance d_{st} will remain the same for each edge st with $st \notin E_i$.

Smoothness. Each $\delta \vec{x}_j$ with $j \notin V_i$ is a smooth function over W_i and satisfies $\delta \vec{x}_j(p_i) = 0$.

Remark. Each $\Gamma_i := (V_i, E_i)$ is a LGT-I, and in particular, Γ_i is rigid. So the shape of each sub-configuration p_i will be preserved by following the displacement.

This section is divided into two parts. In section 3.1, we will prove theorem 2.3.1 and in section 3.2, we will follow theorem 2.3.1 to investigate the infinitesimal version of the displacements.

2.3.2 Proof of theorem 2.3.1

We start by investigating the simplest case where we have only three agents, and they form a nondegenerate triangle in the plane.

Lemma 2.3.2. *Let \vec{x}_1, \vec{x}_2 and \vec{x}_3 form a nondegenerate triangle, i.e, $\vec{x}_3 - \vec{x}_1$ and $\vec{x}_3 - \vec{x}_2$ are linearly independent. There are open neighborhoods U_1 and U_2 of \vec{x}_1 and \vec{x}_2 in \mathbb{R}^2 such that as long as $\vec{x}_1 + \delta\vec{x}_1$ lies in U_1 and $\vec{x}_2 + \delta\vec{x}_2$ lies in U_2 , then there is a unique, smooth function $\delta\vec{x}_3$ defined on $U_1 \times U_2$ such that $|(\vec{x}_3 + \delta\vec{x}_3) - (\vec{x}_i + \delta\vec{x}_i)| = |\vec{x}_3 - \vec{x}_i|$ for $i = 1, 2$ and $\delta\vec{x}_3 = 0$ if $\delta\vec{x}_1 = \delta\vec{x}_2 = 0$.*

Proof. Let $d_{13} := |\vec{x}_3 - \vec{x}_1|$ and $d_{23} := |\vec{x}_3 - \vec{x}_2|$ be two fixed distances. Choose open neighborhoods U_i of \vec{x}_i in \mathbb{R}^2 for each $i = 1, 2, 3$ such that the triangle $p' = (\vec{x}'_1; \vec{x}'_2; \vec{x}'_3)$ is nondegenerate if $\vec{x}'_i \in U_i$ for each $i = 1, 2, 3$. We then define a smooth function ϕ on $\Pi_{i=1}^3 U_i$ by

$$\phi : (\vec{x}'_1, \vec{x}'_2, \vec{x}'_3) \mapsto (|\vec{x}'_3 - \vec{x}'_1|^2 - d_{13}^2, |\vec{x}'_3 - \vec{x}'_2|^2 - d_{23}^2) \quad (2.19)$$

This map is a submersion since the derivative

$$D\phi = \begin{pmatrix} (\vec{x}'_1 - \vec{x}'_3)^T & 0 & (\vec{x}'_3 - \vec{x}'_1)^T \\ 0 & (\vec{x}'_2 - \vec{x}'_3)^T & (\vec{x}'_3 - \vec{x}'_2)^T \end{pmatrix} \quad (2.20)$$

is of full rank at any $p' = (\vec{x}'_1; \vec{x}'_2; \vec{x}'_3) \in \Pi_{i=1}^3 U_i$. The set $\phi^{-1}(0)$ is then a smooth manifold of dimension

4. Shrink U_i , if necessary, so that $\phi^{-1}(0)$ contains only one connected component.

Let $\vec{x}'_1 = (a'_1, b'_1)$ and let $\vec{x}'_2 = (a'_2, b'_2)$. We then define four vectors in \mathbb{R}^6 as

$$\begin{aligned} v_1 &:= (a'_1, 0, 0, 0, 0, 0) \\ v_2 &:= (0, b'_1, 0, 0, 0, 0) \\ v_3 &:= (0, 0, a'_2, 0, 0, 0) \\ v_4 &:= (0, 0, 0, b'_2, 0, 0) \end{aligned} \tag{2.21}$$

The four row vectors, together with the two row vectors in $D\phi$ form a basis of \mathbb{R}^6 , and this holds for any $(\vec{x}'_1; \vec{x}'_2; \vec{x}'_3) \in \Pi_{i=1}^3 U_i$ because $\vec{x}'_3 - \vec{x}'_1$ and $\vec{x}'_3 - \vec{x}'_2$ are linearly independent over $\Pi_{i=1}^3 U_i$. So \vec{x}'_1 and \vec{x}'_2 can be used as coordinates of $\phi^{-1}(0)$. In other words, there is a smooth function $f : U_1 \times U_2 \rightarrow U_3$ such that $(\vec{x}'_1, \vec{x}'_2, f(\vec{x}'_1, \vec{x}'_2)) \in \phi^{-1}(0)$. We then let $\delta\vec{x}_3 := f - \vec{x}_3$, and this completes the proof. \square

We now prove theorem 2.3.1.

Proof of theorem 2.3.1. We prove for the case where p_1 is perturbed. By lemma 2.2.3, we can choose a Henneberg construction with $\Gamma_1 = (V_1, E_1)$ its priority, we then label the vertices of Γ with respect to the order of the construction.

Suppose $V_1 = \{1, \dots, l\}$. Choose open neighborhoods U_i of \vec{x}_i in \mathbb{R}^2 for each $i = 1, \dots, l$, and let $W_1 := \Pi_{i=1}^l U_i$ be the open neighborhood of p_1 in $\mathbb{R}^{2 \times N}$. We may modify the open set W_1 along the proof if necessary. Let $\delta\vec{x}_i \in \mathbb{R}^2$ be a perturbation of \vec{x}_i within U_i for $i = 1, \dots, l$, we now subsequently work out $\delta\vec{x}_{l+1}, \dots, \delta\vec{x}_N$. This is done by induction. Suppose we have found $\delta\vec{x}_{l+1}, \dots, \delta\vec{x}_{m-1}$ with $m \leq N$, and we show there exists a unique displacement $\delta\vec{x}_m$ that does the job. Let s and t be the two vertices that m joins to. There are two situations.

Case I. Suppose the two vertices s, t are both in V_1 . Then by maximality of the canonical partition, we know the triangle formed by \vec{x}_s, \vec{x}_t and \vec{x}_m is nondegenerate because otherwise, vertex m will be contained in V_1 . Lemma 2.3.2 then says that, there is a unique displacement $\delta\vec{x}_m$ of \vec{x}_m , as a smooth function of $\delta\vec{x}_s$ and $\delta\vec{x}_t$, that preserves the values of two distances d_{ms} and d_{mt} as long as the two open neighborhoods U_s and U_t are sufficiently small. Moreover, since $\delta\vec{x}_m = 0$ if $\delta\vec{x}_s = \delta\vec{x}_t = 0$. So given an open neighborhood U_m of \vec{x}_m , we can shrink U_s and U_t , if necessary, so that the vector

$\vec{x}_m + \delta\vec{x}_m$ lies in U_m for any $\delta\vec{x}_s \in U_s$ and any $\delta\vec{x}_t \in U_t$.

Case II. Suppose there is at least one vertex, say s , not contained in V_1 . Consequently the edge st is not in E_1 , so by induction, the distance d_{st} remains unchanged. We may as well assume that the three agents \vec{x}_s , \vec{x}_t and \vec{x}_m are aligned because otherwise, it is covered by case I. We then define

$$\delta\vec{x}_m := \delta\vec{x}_s + \frac{|\vec{x}_m - \vec{x}_s|}{|\vec{x}_t - \vec{x}_s|}(\delta\vec{x}_t - \delta\vec{x}_s) \quad (2.22)$$

the displacement $\delta\vec{x}_m$ of \vec{x}_m will keep the three agents aligned after the displacements, and it is the unique solution that preserves both d_{ms} and d_{mt} . By definition, $\delta\vec{x}_m$ is a linear (hence smooth) function of $\delta\vec{x}_s$ and $\delta\vec{x}_t$, and $\delta\vec{x}_m = 0$ if $\delta\vec{x}_s = \delta\vec{x}_t = 0$. \square

2.3.3 The infinitesimal version of the displacements

In this part, we consider the infinitesimal version of theorem 2.3.1. We will still assume that $\Gamma_i = (V_i, E_i)$, $i = 1, \dots, k$, are subgraphs of Γ with respect to the canonical partition of E , and p_i is the sub-configuration formed by agents \vec{x}_{i_j} with $i_j \in V_i$.

Let W_i be a small open set of p_i in $\mathbb{R}^{2 \times |V_i|}$. We now construct, for each $i = 1, \dots, k$, a smooth function

$$\Delta_i : W_i \rightarrow \mathbb{R}^{2 \times N} \quad (2.23)$$

Let $V_i = (i_1, \dots, i_l)$ be the set of vertices associated with p_i , write $p_i = (\vec{x}_{i_1}, \dots, \vec{x}_{i_l})$. We then define

$$\Delta_i : p_i + (\delta\vec{x}_{i_1}, \dots, \delta\vec{x}_{i_l}) \mapsto p + (\delta\vec{x}_1, \dots, \delta\vec{x}_N) \quad (2.24)$$

Each $\delta\vec{x}_s$, $s \in V_i$, is a perturbation of \vec{x}_s while each $\delta\vec{x}_t$, $t \notin V_i$, is the associated displacement of \vec{x}_t .

We now consider infinitesimal version of the displacements. Let

$$D\Delta_i : TW_i \rightarrow T\mathbb{R}^{2 \times N} \quad (2.25)$$

be the derivative of Δ_i . This map produces an infinitesimal motion of p that will preserve the shapes of p_j for any $j \neq i$, when p_i is perturbed.

To be explicit, let $p_j = (\vec{x}_{j_1}, \dots, \vec{x}_{j_m})$ and let (a_{j_q}, b_{j_q}) be the coordinates of \vec{x}_{j_q} in the plane. We first define three vectors in $\mathbb{R}^{2 \times m}$ as

$$\begin{aligned}\vec{s}_{p_j} &:= (1, 0, 1, 0, \dots, 1, 0)^T \\ \vec{t}_{p_j} &:= (0, 1, 0, 1, \dots, 0, 1)^T \\ \vec{r}_{p_j} &:= (-b_{j_1}, a_{j_1}, -b_{j_2}, a_{j_2}, \dots, -b_{j_m}, a_{j_m})^T\end{aligned}\tag{2.26}$$

Each of the three vectors represents an infinitesimally rigid motion. Vector \vec{s}_{p_j} represents the translations of p_j along a -axis, vector \vec{t}_{p_j} represents the translation of p_j along b -axis and vector \vec{r}_{p_j} represents the counter-clockwise rotation of p_j around the origin. Let L_{p_j} be the subspace in $\mathbb{R}^{2 \times l}$ spanned by \vec{s}_{p_j} , \vec{t}_{p_j} and \vec{r}_{p_j} . We call L_{p_j} **the space of trivial infinitesimal motions** associated with p_j .

Lemma 2.3.3. *Let \vec{v} be a vector in $\mathbb{R}^{2 \times l}$, identified with an element in the tangent space $T_{p_i}W_i \approx \mathbb{R}^{2 \times l}$. Let*

$$\vec{u} := D\Delta_i \vec{v}\tag{2.27}$$

and write

$$\vec{u} = (\vec{u}_1; \dots; \vec{u}_N)\tag{2.28}$$

with each \vec{u}_j a vector in \mathbb{R}^2 that describes the infinitesimal motion of \vec{x}_j in p . Let

$$\vec{u}_{p_j} := (\vec{u}_{j_1}; \dots; \vec{u}_{j_m}) \in \mathbb{R}^{2 \times m}\tag{2.29}$$

be the infinitesimal motion of p_j , then \vec{u}_{p_j} lies in L_{p_j} .

Proof. As the sub-configuration p_j keeps its shape as p_i varies over W_i , the vector \vec{u}_{p_j} has to be contained in L_{p_j} . \square

Remark. : One consequence of lemma 2.3.3 is that

$$\langle \vec{x}_s - \vec{x}_t, \vec{u}_s - \vec{u}_t \rangle = 0 \quad (2.30)$$

for any edge st that is not contained in E_i .

Notice that lemma 2.3.3 holds for any $\vec{v} \in \mathbb{R}^{2 \times l}$, and the result applies to any other sub-configuration that is not p_i . Lemma 2.3.3 will be a useful fact in developing the index/co-index formula.

2.4 Hyperbolicity of a critical orbit

2.4.1 Theorem 2.4.1

Our goal in this section is to introduce theorem 2.4.1 and we will prove part of the statement.

Theorem 2.4.1. *Let n be the dimension of the Euclidean space of agents, and we assume here $n = 2$. Let $\Gamma = (V, E)$ be a LGT-I, and let p be an equilibrium associated with Ψ . Let p_i and Ψ_i , $i = 1, \dots, k$, be sub-configurations and sub-potentials, respectively, with respect to the canonical partition of E associated with p . Then the critical orbit \mathcal{O}_p is hyperbolic if and only if each \mathcal{O}_{p_i} is a hyperbolic critical orbit with respect to Ψ_i .*

In this section, we will only prove the “only if” part, i.e, we show that if the critical orbit \mathcal{O}_p is hyperbolic, then each \mathcal{O}_{p_i} is hyperbolic with respect to Ψ_i . We will leave the proof of the “if” part in the next section. This section is divided into three parts: in section 4.1, we will work out the Hessian of Ψ at an equilibrium p , and establish a necessary and sufficient condition for an critical orbit to be hyperbolic. In section 4.2, we will continue to work out the Hessian of Ψ_i at p_i , and develop a relation between $\mathcal{H}(\Psi)$ and $\mathcal{H}(\Psi_i)$. In section 4.3, we proof the “only if ” part of theorem 2.4.1.

2.4.2 The Hessian of the potential Ψ at an equilibrium p

In this part, we will compute the Hessian matrix of Ψ at an equilibrium p , and give a necessary and sufficient condition for a critical orbit \mathcal{O}_p to be hyperbolic.

Before going on, we define some useful terms, and rearrange entries of p . Let a_i, b_i be the coordinates of agent \vec{x}_i in p . Let \vec{a} and \vec{b} be two vectors in \mathbb{R}^N defined by

$$\begin{aligned}\vec{a} &:= (a_1, \dots, a_N) \\ \vec{b} &:= (b_1, \dots, b_N)\end{aligned}\tag{2.31}$$

We then rearrange entries of p so that

$$p = (\vec{a}; \vec{b})\tag{2.32}$$

We will assume this arrangement in this section.

A matrix is said to be an **interaction matrix** if each column and row has zero-sum. We now define a N -by- N interaction matrix G by specifying each of its off-diagonal entries. Let G_{ij} , $i \neq j$ be the ij -th entry of G , then define

$$G_{ij} = \begin{cases} g_{ij}(d_{ij}) & ij \in E \\ 0 & \text{otherwise} \end{cases}\tag{2.33}$$

The gradient vector field associated with Ψ then can be expressed as

$$\dot{p} = \text{Diag}(G, G)p\tag{2.34}$$

where $\text{Diag}(G, G)$ is a $2N$ -by- $2N$ block-diagonal matrix.

Let p be an equilibrium, we now work out an explicit form of the Hessian $\mathcal{H}(\Psi)$ at p . First we define three N -by- N interaction matrices. $D^{a,a}$, $D^{a,b}$ and $D^{b,b}$. Each is defined by specifying its

off-diagonal entries.

$$D_{ij}^{a,a} = \begin{cases} g'_{ij}(d_{ij})(a_i - a_j)^2/d_{ij} & ij \in E \\ 0 & \text{otherwise} \end{cases} \quad (2.35)$$

$$D_{ij}^{a,b} = \begin{cases} g'_{ij}(d_{ij})(a_i - a_j)(b_i - b_j)/d_{ij} & ij \in E \\ 0 & \text{otherwise} \end{cases} \quad (2.36)$$

$$D_{ij}^{b,b} = \begin{cases} g'_{ij}(d_{ij})(b_i - b_j)^2/d_{ij} & ij \in E \\ 0 & \text{otherwise} \end{cases} \quad (2.37)$$

Then the Hessian $\mathcal{H}(\Psi)$ at p is given by

$$\mathcal{H}_p(\Psi) = \begin{pmatrix} D^{a,a} + G & D^{a,b} \\ D^{a,b} & D^{b,b} + G \end{pmatrix} \quad (2.38)$$

Lemma 2.4.2. *Let p be an equilibrium associated with Ψ . If $p' \in \mathcal{O}_p$, then $\mathcal{H}_{p'}(\Psi)$ and $\mathcal{H}_p(\Psi)$ share the same set of eigenvalues.*

Proof. Let $\alpha = t_{\vec{v}} \cdot \theta$ be an element in $SE(2)$ with $\vec{v} = (v_a, v_b)$, and we assume $p' = \alpha \cdot p$. After re-arranging entries of p , the group action is given by

$$\alpha \cdot p = (\theta \otimes I_N)p + (v_a \vec{e}; v_b \vec{e}) \quad (2.39)$$

where I_N is the N -by- N identity matrix, \otimes is the Kronecker product and \vec{e} is a vector of all ones in \mathbb{R}^N . Then

$$\mathcal{H}_{p'}(\Psi) = (\theta \otimes I_N) \mathcal{H}_p(\Psi) (\theta \otimes I_N)^T \quad (2.40)$$

If v is an eigenvector of $\mathcal{H}_p(\Psi)$, then $(\theta \otimes I_N)v$ will be an eigenvector of $\mathcal{H}_{p'}(\Psi)$ with respect to the same eigenvalue. This is because $(\theta \otimes I_N)^T(\theta \otimes I_N) = I_{2N}$. \square

Corollary 2.4.3. *Let p be an equilibrium associated with Ψ . Let $L_p \subset \mathbb{R}^{2 \times N}$ be the space of trivial*

infinitesimal motions. As a consequence of the rearrangement of entries of p , the subspace L_p is spanned by

$$\begin{aligned}\vec{s} &= (\vec{e}, 0) \\ \vec{t} &= (0, \vec{e}) \\ \vec{r} &= (-\vec{b}, \vec{a})\end{aligned}\tag{2.41}$$

The orbit \mathcal{O}_p is hyperbolic if and only if the kernel of $\mathcal{H}_p(\Psi)$ is L_p .

Proof. First notice that L_p lies in the kernel of $\mathcal{H}_p(\Psi)$ because L_p can be identified with the tangent space $T_p\mathcal{O}_p$. So the Hessian $\mathcal{H}(\Psi)$ is nondegenerate when restricted to the normal space $N_p\mathcal{O}_p$ at p if and only if the kernel of $\mathcal{H}_p(\Psi)$ is L_p . Then by lemma 2.4.2, it is also equivalent that the critical orbit \mathcal{O}_p is hyperbolic. \square

2.4.3 The Hessian of a sub-potential Ψ_i at a sub-configuration p_i

Let p be an equilibrium associated with Ψ , and let p_i and Ψ_i , $i = 1, \dots, k$, be sub-configurations and sub-potentials, respectively, with respect to the canonical partition of E associated with p . We have showed in section 2.2 that p_i is an equilibrium associated with Ψ_i . We will now work out the Hessian of Ψ_i at p_i .

Observation 1. The Hessian $\mathcal{H}(\Psi_i)$ of Ψ_i at a sub-configuration p_i can be derived in three steps.

Step 1. If $j \notin V_i$, then remove the j -th column/row from each of the four matrices G , $D^{a,a}$, $D^{a,b}$ and $D^{b,b}$. This trims the size of the four matrices down to $|V_i|$ -by- $|V_i|$.

Step 2. Keep values of all off-diagonal entries for each of the four trimmed matrices, but change values of diagonal entries so that each matrix becomes an interaction matrix. We let G_i , $D_i^{a,a}$, $D_i^{a,b}$ and $D_i^{b,b}$ be the four modified interaction matrices.

Step 3. The Hessian $\mathcal{H}_{p_i}(\Psi_i)$ is then given by

$$\mathcal{H}_{p_i}(\Psi_i) = \begin{pmatrix} D_i^{a,a} + G_i & D_i^{a,b} \\ D_i^{a,b} & D_i^{b,b} + G_i \end{pmatrix}\tag{2.42}$$

In other words, the Hessian $\mathcal{H}_{p_i}(\Psi_i)$ can be understood as if the agents in p_i formed an isolated system.

There is actually a simple algebraic relation between $\mathcal{H}_p(\Psi)$ and $\mathcal{H}_{p_i}(\Psi_i)$. To see this, we define for each $i = 1, \dots, k$ a $2N$ -by- $2N$ interaction matrix H_i . This is done by three steps.

Step 1. If $j \notin V_i$, then we set the j -th column/row to zero for each of the four matrices G , $D^{a,a}$, $D^{a,b}$ and $D^{b,b}$.

Step 2. Keep values of all off-diagonal entries for each of the four modified matrices, but change values of diagonal entries so that each matrix becomes an interaction matrix. We let \hat{G}_i , $\hat{D}_i^{a,a}$, $\hat{D}_i^{a,b}$ and $\hat{D}_i^{b,b}$ be the four modified interaction matrices.

Step 3. The matrix H_i is then defined by

$$H_i := \begin{pmatrix} \hat{D}_i^{a,a} + \hat{G}_i & \hat{D}_i^{a,b} \\ \hat{D}_i^{a,b} & \hat{D}_i^{b,b} + \hat{G}_i \end{pmatrix} \quad (2.43)$$

So each H_i is an expansion of $\mathcal{H}_{p_i}(\Psi_i)$ in an appropriate way by filling with zeros. An equation that relates $\mathcal{H}_p(\Psi)$ to H_i is given by

$$\mathcal{H}_p(\Psi) = \sum_{i=1}^k H_i \quad (2.44)$$

This equation will be useful in the proof of theorem 2.4.1 and in the development of the index/co-index formula.

2.4.4 Proof of the “only if” part of theorem 2.4.1

Before going on, we define some useful notations as we will use in the proof. Let $\vec{v} = (v_1, \dots, v_N)$ be a vector in \mathbb{R}^N , and let p_i be a sub-configuration of p formed by agents $\vec{x}_{i_1}, \dots, \vec{x}_{i_l}$ with $i_1 < \dots < i_l$, we then define a vector \vec{v}_{p_i} in \mathbb{R}^l by restricting \vec{v} to p_i , i.e.,

$$\vec{v}_{p_i} := (v_{i_1}, \dots, v_{i_l}) \quad (2.45)$$

If $\vec{u} = (\vec{v}, \vec{w})$ is a vector in $\mathbb{R}^{2 \times N}$ with \vec{v} and \vec{w} two vectors in \mathbb{R}^N . We then define a vector in $\mathbb{R}^{2 \times l}$ by

$$\vec{u}_{p_i} := (\vec{v}_{p_i}, \vec{w}_{p_i}) \quad (2.46)$$

Let \vec{s} , \vec{t} and \vec{r} be the three vectors in $\mathbb{R}^{2 \times N}$ defined by equation (2.41). Let L_{p_i} be the space of trivial infinitesimal motions of p_i , then L_{p_i} is spanned by \vec{s}_{p_i} , \vec{t}_{p_i} and \vec{r}_{p_i} . Notice that this notation will agree with our earlier definition of \vec{s}_{p_j} , \vec{t}_{p_j} and \vec{r}_{p_j} in equation (2.26) after we rearrange entries of the three vectors.

In the proof, we will use the map Δ_i defined in equation (2.24), and we now modify the map with respect to the rearrangement of entries. Let $\delta \vec{x}_s := (\delta a_s, \delta b_s)$ and let

$$\begin{aligned} \delta \vec{a} &:= (\delta a_1, \dots, \delta a_N) \\ \delta \vec{b} &:= (\delta b_1, \dots, \delta b_N) \end{aligned} \quad (2.47)$$

Let $\delta \vec{a}_{p_i}$ and $\delta \vec{b}_{p_i}$ be defined with respect to our notation, and let

$$\begin{aligned} \delta p &:= (\delta \vec{a}, \delta \vec{b}) \\ \delta p_i &= (\delta \vec{a}_{p_i}, \delta \vec{b}_{p_i}) \end{aligned} \quad (2.48)$$

The map Δ_i is then modified to be

$$\Delta_i : p_i + \delta p_i \mapsto p + \delta p \quad (2.49)$$

The derivative $D\Delta_i$ changes correspondingly.

We are now ready to prove theorem 2.4.1.

Proof of theorem 2.4.1. We show that if there is a sub-configuration, say p_1 , of p such that the critical orbit \mathcal{O}_{p_1} is not hyperbolic with respect to Ψ_1 , then \mathcal{O}_p won't be hyperbolic with respect to Ψ . Suppose p_1 is a sub-configuration of l agents, we then choose a vector $\vec{v} \in \mathbb{R}^{2 \times l}$ such that \vec{v} is contained in the kernel of \mathcal{H}_{p_1} , but not contained in L_{p_1} . Let $\vec{u} := D\Delta_1 \vec{v}$, we will now show that \vec{u} is contained in the

kernel of $\mathcal{H}_p(\Psi)$, but not contained in L_p . Notice that if this the case, we will then establish theorem 2.4.1 by applying corollary 2.4.3.

Proof that $\vec{u} \notin L_p$. This holds because if $\vec{u} \in L_p$, then necessarily \vec{u}_{p_1} lies in L_{p_1} . On the other hand, we have $\vec{u}_{p_1} = \vec{v}$, and \vec{v} is chosen to be perpendicular to L_{p_1} , so then \vec{u} is not contained in L_p .

Proof that $\mathcal{H}_p(\Psi)\vec{u} = 0$. By equation (2.44), we have

$$\mathcal{H}_p(\Psi)u = \sum_{i=1}^k H_i u \quad (2.50)$$

so it suffices to show that $H_i \vec{u} = 0$ for each $i = 1, \dots, k$. First notice that $H_i \vec{u} = 0$ if and only if $\mathcal{H}_{p_i}(\Psi_i) \vec{u}_{p_i} = 0$ because each H_i , by its way of construction, is an appropriate expansion of $\mathcal{H}_{p_i}(\Psi_i)$ by filling with zeros. By our choice of v , we have $\mathcal{H}_{p_1}(\Psi_1)u_{p_1} = 0$.

So we need to show that for each $i = 2, \dots, k$, we have $\mathcal{H}_{p_i}(\Psi_i)u_{p_i} = 0$. The Hessian $\mathcal{H}_{p_i}(\Psi_i)$ at p_i is given by

$$\mathcal{H}_{p_i}(\Psi_i) = \begin{pmatrix} D_i^{a,a} & D_i^{a,b} \\ D_i^{a,b} & D_i^{b,b} \end{pmatrix} + \begin{pmatrix} G_i & 0 \\ 0 & G_i \end{pmatrix} \quad (2.51)$$

Now fix an $i = 2, \dots, k$, and we assume that the set of vertices of Γ_i is $V_i = \{i_1, \dots, i_m\}$ with $i_1 < \dots < i_m$. So then \vec{u}_{p_i} is in $\mathbb{R}^{2 \times m}$, and we write

$$\vec{u}_{p_i} = (\vec{y}, \vec{z}) \quad (2.52)$$

with \vec{y} and \vec{z} two vectors in \mathbb{R}^m . It suffices for us to show that

$$G_i \vec{y} = G_i \vec{z} = 0 \quad (2.53)$$

$$D_i^{a,a} \vec{y} + D_i^{a,b} \vec{z} = D_i^{a,b} \vec{y} + D_i^{b,b} \vec{z} = 0 \quad (2.54)$$

as they are combined to establish the equality $\mathcal{H}_{p_i}(\Psi_i)u_{p_i} = 0$

We first prove equality (2.53). By lemma 2.3.3, the vector \vec{u}_{p_i} lies in L_{p_i} , so the two vectors \vec{y}

and \vec{z} satisfy

$$\begin{aligned}\vec{y} &= c_1 \vec{b}_{p_i} + c_2 \vec{e}_{p_i} \\ \vec{z} &= -c_1 \vec{a}_{p_i} + c_3 \vec{e}_{p_i}\end{aligned}\tag{2.55}$$

where c_1 , c_2 and c_3 are scalar coefficients and \vec{e}_{p_i} , by notation, is a vector of all ones in \mathbb{R}^m . On the other hand, p_i is an equilibrium associated with the potential Ψ_i , so $G_i a_{p_i} = G_i b_{p_i} = 0$. Moreover, since G_i is an interaction matrix, $G_i \vec{e}_{p_i} = 0$. This then establishes equation (2.53).

We now prove equality (2.54). We will only prove $D_i^{a,a} \vec{y} + D_i^{a,b} \vec{z} = 0$ as the other follows in the same way. Write $\vec{y} = (y_1, \dots, y_m)$ and $\vec{z} = (z_1, \dots, z_m)$, and for each $j = 1, \dots, m$, we let

$$\vec{w}_j := (y_j, z_j)\tag{2.56}$$

By unravelling the definition of \vec{y} and \vec{z} and the map $D\Delta_i$, we see that each \vec{w}_j is actually the infinitesimal motion of agent \vec{x}_{i_j} of p_i with respect to the infinitesimal perturbation \vec{v} of p_1 . Let $[D_i^{a,a} \vec{y} + D_i^{a,b} \vec{z}]_s$ be the s -th entry of $D_i^{a,a} \vec{y} + D_i^{a,b} \vec{z}$, and let

$$\tau_{st} := \begin{cases} g'_{ij}(d_{i_s i_t})(a_{i_s} - a_{i_t})/d_{i_s i_t} & i_s i_t \in E_i \\ 0 & \text{otherwise} \end{cases}\tag{2.57}$$

then we have

$$[D_i^{a,a} \vec{y} + D_i^{a,b} \vec{z}]_s = \sum_{t=1}^m \tau_{st} \langle \vec{x}_{i_s} - \vec{x}_{i_t}, \vec{w}_s - \vec{w}_t \rangle\tag{2.58}$$

Since E_i is disjoint with E_1 , for each $i_s i_t \in E_i$, we have

$$\langle \vec{x}_{i_s} - \vec{x}_{i_t}, \vec{w}_s - \vec{w}_t \rangle = 0\tag{2.59}$$

This is a consequence of lemma 2.3.3, as we have established in equation (2.30). So the vector $D_i^{a,a} \vec{y} + D_i^{a,b} \vec{z}$ does vanish. The other vector $D_i^{b,a} \vec{y} + D_i^{b,b} \vec{z}$ vanishes by the same reason. In fact, if we replace $(a_{i_s} - a_{i_t})$ with $(b_{i_s} - b_{i_t})$ when defining τ_{st} in equation (2.57), then $[D_i^{a,b} \vec{y} + D_i^{b,b} \vec{z}]_s$ takes the same formula described by equation (2.58). This then establishes theorem 2.4.1. \square

2.5 The index/co-index formula

Our goal in this section is to establish the index/co-index formula, and the “if” part of theorem 2.4.1.

Theorem 2.5.1 (Index/Co-index Formula). *Let n be the dimension of the Euclidean space of agents, and we suppose $n = 2$. Let $\Gamma = (V, E)$ be a LGT-I, and let p be an equilibrium associated with Ψ . Let p_i and Ψ_i , $i = 1, \dots, k$, be sub-configurations and sub-potentials, respectively, with respect to the canonical partition of E associated with p . Let $\iota_-(\mathcal{O}_{p_i})/\iota_+(\mathcal{O}_{p_i})$ be the index/co-index of \mathcal{O}_{p_i} with respect to Ψ_i , then the index/co-index of \mathcal{O}_p with respect to Ψ is given by*

$$\begin{aligned}\iota_-(\mathcal{O}_p) &= \sum_{i=1}^k \iota_-(\mathcal{O}_{p_i}) \\ \iota_+(\mathcal{O}_p) &= \sum_{i=1}^k \iota_+(\mathcal{O}_{p_i})\end{aligned}$$

This theorem will be developed after a sequence of lemmas and corollaries. We will still assume in this section the rearrangement of entries of a configuration p as we defined in the beginning of the previous section. We start by introducing the inertia of a real, symmetric matrix.

Negative/Positive index of inertia of a real symmetric matrix. Let H be m -by- m a real symmetric matrix, let n_-/n_+ be the number negative/positive eigenvalues of H , then n_-/n_+ is called the negative/positive index of inertia of H . Let n_0 be the number of zero eigenvalues, then $n_- + n_+ + n_0 = m$.

Let p be an equilibrium, in our case, the Morse index/co-index of \mathcal{O}_p is just the negative/positive index of inertia of the Hessian matrix $\mathcal{H}_p(\Psi)$. We now state the Sylvester’s law of inertia.

Fact 2.5.2 (Sylvester’s law of inertia). *Let H be a m -by- m real symmetric matrix. Let A be a m -by- m nonsingular matrix, then the matrix $A^T H A$ share the same negative/positive index of inertia with H .*

A complete proof of fact 2.5.2 can be found in [17]. Here is a corollary of lemma 2.5.2.

Corollary 2.5.3. *Let p be an equilibrium associate with Ψ . Let A be a $(2N)$ -by- $(2N-3)$ matrix, and we define a $(2N-3)$ -by- $(2N-3)$ matrix by*

$$H_A := A^T \mathcal{H}_p(\Psi) A \quad (2.60)$$

If H_A is non-singular, then the critical orbit \mathcal{O}_p is hyperbolic. Moreover, the Morse index/co-index of \mathcal{O}_p agrees with the negative/positive index of inertia of H_A .

Proof. Let $\vec{u}_i \in \mathbb{R}^{2 \times N}$, $i = 1, \dots, 2N-3$, be the i -th column of matrix A . Clearly these $(2N-3)$ vectors are linearly independent. On the other hand, each \vec{u}_i has to lie outside L_p because otherwise $\mathcal{H}_p(\Psi)\vec{u}_i = 0$ and hence, H_A is singular. So the set of vectors \vec{u}_i , $i = 1, \dots, 2N-3$, together with the three vectors \vec{s} , \vec{t} and \vec{r} in L_p as we defined in equation (2.41) form a basis for $\mathbb{R}^{2 \times N}$. Let \tilde{A} be a $2N$ -by- $2N$ matrix defined by

$$\tilde{A} = (\vec{u}_1, \dots, \vec{u}_{2N-3}, \vec{r}, \vec{s}, \vec{t}) \quad (2.61)$$

then $\tilde{A}^T \mathcal{H}_p(\Psi) \tilde{A} = \text{Diag}(H_A, 0_{3 \times 3})$. Since H_A is nonsingular, by lemma 2.5.2 the Hessian matrix $\mathcal{H}_p(\Psi)$ has only three zero eigenvalues with L_p the corresponding eigenspace. So the critical orbit \mathcal{O}_p is hyperbolic by corollary 2.4.3, and the matrices $\mathcal{H}_p(\Psi)$ and H_A share the same negative/positive index of inertia by Sylvester's law of inertia. \square

We now follow the idea of corollary 2.5.3, and choose a particular set of vectors $\vec{u}_1, \dots, \vec{u}_{2N-3}$ in \mathbb{R}^N so that the matrix H_A is a nonsingular, diagonal matrix, and each diagonal entry is a nonzero eigenvalue of the Hessian matrix $\mathcal{H}_{p_i}(\Psi_i)$ for some $i = 1, \dots, k$. First we show

Lemma 2.5.4. *Suppose the dimension of the Euclidean space of agents is two. Let $\Gamma = (V, E)$ be a LGT-I, and let p be an equilibrium associated with Ψ . Let p_i and Ψ_i , $i = 1, \dots, k$, be sub-configurations and sub-potentials, respectively, with respect to the canonical partition of E associated with p . Let l_i be the number of nonzero eigenvalues of the Hessian matrix $\mathcal{H}_{p_i}(\Psi_i)$. If the critical orbit \mathcal{O}_p is hyperbolic, then $\sum_{i=1}^k l_i = 2N-3$.*

Proof. Let $\Gamma_i = (V_i, E_i)$, $i = 1, \dots, k$, be subgraphs of Γ with respect to the canonical partition of E

associated with p . Since each subgraph Γ_i is a LGT-I, we have $|E_i| = 2|V_i| - 3$. On the other hand, if \mathcal{O}_p is hyperbolic with respect to Ψ , then each \mathcal{O}_{p_i} is hyperbolic with respect to Ψ_i . So

$$l_i = 2|V_i| - 3 = |E_i| \quad (2.62)$$

this then implies that $\sum_{i=1}^k l_i = \sum_{i=1}^k |E_i| = |E| = 2N - 3$. \square

So suggested by lemma 2.5.4, we will now construct a vector \vec{u}_{i_j} for each nonzero eigenvalue λ_j of the Hessian matrix $\mathcal{H}_{p_i}(\Psi_i)$.

Lemma 2.5.5. *Suppose the dimension of the Euclidean space of agents is two. Let $\Gamma = (V, E)$ be a LGT-I, and let \mathcal{O}_p be a hyperbolic critical orbit with respect to Ψ . Let p_i and Ψ_i , $i = 1, \dots, k$, be sub-configurations and sub-potentials, respectively, with respect to the canonical partition of E . Assume $|E_i| = l_i$, and we let $\lambda_{i_1}, \dots, \lambda_{i_{l_i}}$ be the nonzero eigenvalues of the Hessian matrix $\mathcal{H}_{p_i}(\Psi_i)$. Let $\vec{v}_{i_s} \in \mathbb{R}^{2 \times l}$ be the unit-length eigenvector of $\mathcal{H}_{p_i}(\Psi_i)$ with respect to λ_{i_s} . Define*

$$\vec{u}_{i_s} := D\Delta_i \vec{v}_{i_s} \quad (2.63)$$

then

$$\langle \vec{u}_{i_s}, \mathcal{H}_p(\Psi) \vec{u}_{j_t} \rangle = \delta_{ij} \delta_{st} \lambda_{i_s} \quad (2.64)$$

where δ is the Kronecker delta, i.e, $\delta_{ij} = 1$ if $i = j$, otherwise, $\delta_{ij} = 0$.

Proof. We first fix an i , and show that

$$\langle \vec{u}_{i_s}, \mathcal{H}_p(\Psi) \vec{u}_{i_t} \rangle = \delta_{st} \lambda_{i_s} \quad (2.65)$$

Recall the summation $\mathcal{H}_p(\Psi) = \sum_{j=1}^k H_j$ established in equation (2.44). We then apply the same arguments, as we used to prove theorem 2.4.1 in section 4.3, to conclude that if $i \neq j$, then

$$H_j \vec{u}_{i_s} = 0 \quad (2.66)$$

so we have

$$\mathcal{H}_p(\Psi)\vec{u}_{i_s} = \sum_{j=1}^k H_j \vec{u}_{i_s} = H_i \vec{u}_{i_s} \quad (2.67)$$

Since H_i is an expansion of $\mathcal{H}_{p_i}(\Psi_i)$ by filling with zeros, and the restriction of \vec{u}_{i_s} to p_i is just v_{i_s} , we then have

$$\langle \vec{u}_{i_s}, H_i \vec{u}_{i_t} \rangle = \langle \vec{v}_{i_s}, \mathcal{H}_{p_i}(\Psi_i) \vec{v}_{i_t} \rangle = \delta_{st} \lambda_{i_s} \quad (2.68)$$

It now remains to show that if $i \neq j$, then

$$\langle \vec{u}_{j_s}, \mathcal{H}_p(\Psi) \vec{u}_{i_t} \rangle = 0 \quad (2.69)$$

We now use equation (2.67) again and get

$$\langle \vec{u}_{j_s}, \mathcal{H}_p(\Psi) \vec{u}_{i_t} \rangle = \langle \vec{u}_{j_s}, H_i \vec{u}_{i_t} \rangle = \langle \vec{u}_{i_t}, H_i \vec{u}_{j_s} \rangle = 0 \quad (2.70)$$

This then establishes the lemma. \square

We are now ready to prove theorem 2.5.1.

Proof of theorem 2.5.1. Let $\Gamma_i = (V_i, E_i)$, $i = 1, \dots, k$, be subgraphs of Γ with respect to the canonical partition of E associated with p . Let \vec{u}_{i_j} , $1 \leq j \leq l_i$ and $1 \leq i \leq k$, be vectors in $\mathbb{R}^{2 \times N}$ defined in the statement of lemma 2.5.5. Since each \vec{u}_{i_j} one-to-one corresponds to a nonzero eigenvalue λ_{i_j} of a Hessian matrix $\mathcal{H}_{p_i}(\Psi_i)$, so by lemma 2.5.4, the total number of vectors \vec{u}_{i_j} , $1 \leq j \leq l_i$ and $1 \leq i \leq k$, is $(2N - 3)$. Let A be a $(2N)$ -by- $(2N - 3)$ matrix formed by arranging \vec{u}_{i_j} , $1 \leq j \leq l_i$ and $1 \leq i \leq k$, as column vectors, i.e.,

$$A := (\vec{u}_{1_1}, \vec{u}_{1_2} \cdots, \vec{u}_{k_{l_k}}) \quad (2.71)$$

So then, by lemma 2.5.5

$$A^T \mathcal{H}_p(\Psi) A = \text{diag}(\lambda_{1_1}, \lambda_{1_2} \cdots, \lambda_{k_{l_k}}) \quad (2.72)$$

At this moment, it is clear that the negative/positive inertia of $A^T \mathcal{H}_p(\Psi) A$ is equal $\sum_{i=1}^k \iota_{-}(\mathcal{O}_{p_i}) /$

$\sum_{i=1}^k \iota_+(\mathcal{O}_{p_i})$. Then by corollary 2.5.3, we conclude that

$$\begin{aligned}\iota_-(\mathcal{O}_p) &= \sum_{i=1}^k \iota_-(\mathcal{O}_{p_i}) \\ \iota_+(\mathcal{O}_p) &= \sum_{i=1}^k \iota_+(\mathcal{O}_{p_i})\end{aligned}$$

This then establishes the index/co-index formula. \square

Remark (I). This proof also establishes the “if” part of theorem 2.4.1. Because if each critical orbit \mathcal{O}_{p_i} is hyperbolic with respect to Ψ_i , we then actually find the matrix A defined by equation (2.71) with $A^T \mathcal{H}_p(\Psi) A$ a nonsingular, diagonal matrix. Then we apply corollary 2.5.3 to conclude that the critical orbit \mathcal{O}_p is hyperbolic with respect to Ψ .

Remark (II). The index/co-index formulae also holds for the case where the critical orbit \mathcal{O}_p is not hyperbolic, and the proof uses the same arguments as we established in this section.

At this moment, we have established the main theorem of this chapter. We end this section with a corollary of theorem 2.5.1. This corollary is an application of the index/co-index formula on a spacial class of configurations, as described below.

Strongly non-degenerate configuration. Let $\Gamma = (V, E)$ be a LGT-I, let p be a planar configuration and let $\sigma = (E_1, \dots, E_k)$ be the canonical partition of E associated with E . We say a configuration p is strongly non-degenerate if each subset E_i is a singleton. In other words, we choose a Henneberg construction of Γ , and label the vertices of Γ with respect to the order of the construction. For each vertex $i = 3, \dots, N$, we let i^1 and i^2 be the two vertices i joins to. Then the three agents \vec{x}_{i^1} , \vec{x}_{i^2} and \vec{x}_i are not aligned for each $i = 3, \dots, N$. Notice that this holds for any Henneberg construction of Γ by lemma 2.2.1.

Corollary 2.5.6. *Let n be the dimension of the Euclidean space of agents, and suppose $n = 2$. Let $\Gamma = (V, E)$ be a LGT-I, and let p be an equilibrium associated with Ψ . If p is strongly non-degenerate, then $g_{ij}(d_{ij}) = 0$ for all $ij \in E$. The critical orbit \mathcal{O}_p is hyperbolic if and only if $g'_{ij}(d_{ij}) \neq 0$ for all*

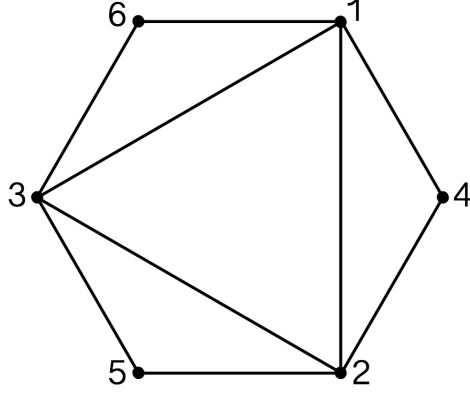


Figure 11: A strongly nondegenerate configuration. The order of the vertices is with respect to a Henneberg construction, and by following this Henneberg construction, the canonical partition completely decomposes E as union of singletons.

$ij \in E$. We now define

$$\begin{aligned} E^s &:= \{ij \in E | g'(d_{ij}) > 0\} \\ E^u &:= \{ij \in E | g'(d_{ij}) < 0\} \end{aligned} \tag{2.73}$$

then the index and the co-index of the orbit \mathcal{O}_p are $|E^s|$ and $|E^u|$, respectively.

Proof. Since p is strongly nondegenerate, the set of edges decomposes completely to singletons with respect to the canonical partition. By theorem 2.1, each two-agent sub-configuration p_{ij} formed by \vec{x}_i and \vec{x}_j , $ij \in E$, is an equilibrium associated with the potential function Ψ_{ij} where

$$\Psi_{ij}(\vec{v}, \vec{u}) := \int_1^{|\vec{v}-\vec{u}|} x g_{ij}(x) dx \tag{2.74}$$

so we must have $g(d_{ij}) = 0$.

We now work out the eigenvalues of the Hessian matrix $\mathcal{H}_{p_{ij}}(\Psi_{ij})$. By lemma 2.4.2, the set of eigenvalues is invariant under the group action of rigid motion. So we may assume that $\vec{x}_i = (a_i, 0)$ and $\vec{x} = (0, 0)$, then

$$\mathcal{H}_{p_{ij}}(\Psi_{ij}) = \begin{pmatrix} H^{a,a} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.75}$$

with

$$H^{a,a} := g'_{ij}(|a_i|)a_i^2 \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2.76)$$

So $\mathcal{H}_{p_{ij}}(\Psi_{ij})$ has at most one nonzero eigenvalue as $-2g'_{ij}(|a_i|)a_i^2$. So the critical orbit $\mathcal{O}_{p_{ij}}$ is hyperbolic with respect to Ψ_{ij} if and only if $g'_{ij}(|a_i|) \neq 0$. The Morse index of $\mathcal{O}_{p_{ij}}$ with respect to Ψ_{ij} is 1 if $g'_{ij}(|a_i|) > 0$, and 0 if $g'_{ij}(|a_i|) < 0$. So we have $\iota_-(\mathcal{O}_p) = |E^s|$, and similarly $\iota_+(\mathcal{O}_p) = |E^u|$. \square

We here mention a fact without proof that if we require each interaction function g_{ij} , $ij \in E$, have only one zero d_{ij}^* and $g'(d_{ij}^*) > 0$, e.g, consider the class of rational functions

$$g_{ij}(d) = -\frac{\sigma_1}{d^{n_1}} + \frac{\sigma_2}{d^{n_2}} \quad (2.77)$$

with $\sigma_i > 0$ for $i = 1, 2$ and $n_1 > n_2 > 1$. Then all critical orbits of non-degenerate configurations are stable. Moreover, by perturbing $g_{ij}(d)$, $ij \in E$, if necessarily, critical orbits of nondegenerate configurations are the only stable orbits, and there are as many as 2^{N-2} of them.

Chapter 3

Genericity of Equivariant Morse Functions

3.1 Definitions and main theorem

Equivariant Morse theory has proved to be a useful tool for studying reciprocal multi-agent (RMA) systems. For the theory to work, we have to take the assumption that the associated potential function is an equivariant Morse function, yet it is still an open question whether the potential function associated with a RMA system is generically an equivariant Morse function? We here quote from [2] by Anderson and Helmke, in the conclusion they wrote

an open problem is thus to give sufficient conditions on a graph such that the associated potential functions are generically equivariant Morse functions. At this point it is not even known whether the genericity assumption is satisfied for complete graphs.

In this chapter, we will investigate this question under the assumption that the network topology Γ is a complete graph, but we sharpen our result by requiring all interactions functions g_{ij} , $i < j$, be identical. So the equations of motion, for agents $\vec{x}_1, \dots, \vec{x}_N \in \mathbb{R}^n$ with $N > n$, take the form

$$\dot{\vec{x}}_i = \sum_{j=1, j \neq i}^N g(d_{ij})(\vec{x}_j - \vec{x}_i), \quad i = 1, \dots, N \quad (3.1)$$

We assume here g is a C^k -function with $k \geq 1$, i.e, derivatives $g', \dots, g^{(k)}$ exist and are continuous, and g satisfies conditions of *strong repulsion* and *fading attraction*.

Strong repulsion. $\lim_{d \rightarrow 0} dg(d) = -\infty$ and $\lim_{d \rightarrow 0} \int_d^1 xg(x) = -\infty$.

Fading attraction. $g(d) > 0$ if $d \gg 1$ and $\lim_{d \rightarrow \infty} dg(d) = 0$

Consequently the potential function associated with system (3.1) is given by

$$\Psi_g(\vec{x}_1, \dots, \vec{x}_N) := \sum_{i < j} \int_1^{d_{ij}} xg(x)dx \quad (3.2)$$

The subindex g reminds us that Ψ_g is generated by the scalar function g .

In this chapter, we only consider configurations with zero centroids, so the configuration space P is given by

$$P := \{(\vec{x}_1; \dots; \vec{x}_N) \mid \sum_{i=1}^N \vec{x}_i = 0 \text{ and } \vec{x}_i \neq \vec{x}_j, \forall i \neq j\} \quad (3.3)$$

Since the effect of translation is eliminated by requiring the centroid of a configuration be zero, the group action of rigid motion on P is then reduced to the $SO(n)$ -action, i.e, we send $\theta \in SO(n)$ and $p = (\vec{x}_1, \dots, \vec{x}_N)$ to

$$\theta \cdot p := (\theta \vec{x}_1; \dots; \theta \vec{x}_N) \quad (3.4)$$

We will still let \mathcal{O}_p be the orbit of p with respect to the $SO(n)$ -action.

A subset $K \subset P$ is said to be $SO(n)$ -**invariant** if $\theta \cdot K = K$ for any $\theta \in SO(n)$. In other words, a $SO(n)$ -invariant subset K is a union of orbits \mathcal{O}_p in P . At the end of chapter II, we have defined the notion of *strongly nondegenerate* configurations, a less restrictive version is the notion of *nondegenerate* configurations, as we define now. Let p be a configuration in P , the **rank** of p is defined to be the dimension of the subspace in \mathbb{R}^n spanned by $\{\vec{x}_i - \vec{x}_1, \dots, \vec{x}_i - \vec{x}_N\}$ for some $i = 1, \dots, N$. The definition is, in fact, independent of the choice of index i . A geometric interpretation of the rank is the least dimension of a subspace in \mathbb{R}^n that embeds p . If the rank of p is n , then p is said to be **nondegenerate**, otherwise p is said to be **degenerate**. Since we have required the number of agents be greater than the dimension of the Euclidean space of agents, non-degenerate configurations are actually open and dense in P with respect to the normal Euclidean topology. We also notice that the space of nondegenerate configuration is $SO(n)$ -invariant because if p is nondegenerate, then so is any configuration in its orbit \mathcal{O}_p .

We will now introduce **the space of interaction functions**. Let \mathcal{G} be the set of all C^k , $1 \leq k \leq +\infty$, functions from \mathbb{R}^+ to \mathbb{R} that satisfy conditions of *strong repulsion* and *fading attraction*, and we equip \mathcal{G} with the Whitney C^k -topology. We describe the Whitney C^k -topology on \mathcal{G} by defining a basis of open sets. Let g be a function in \mathcal{G} , and we define an open ball $B_\delta(g)$ in \mathcal{G} by

$$B_\delta(g) := \{\tilde{g} \in \mathcal{G} \mid \forall d \in \mathbb{R}^+, \sup_{0 \leq i \leq k} |\tilde{g}^{(i)}(d) - g^{(i)}(d)| < \delta(d)\} \quad (3.5)$$

where $\delta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function. By varying g in \mathcal{G} and δ in $C^0(\mathbb{R}^+)$, we then get a basis of open sets for the Whitney C^k -topology on \mathcal{G} . In its most general form, the Whitney C^k -topology is defined via jet bundles, and we refer readers to the second chapter of [18] for detail.

Let K be a $SO(n)$ -invariant subset in P , and we let $\Psi_g|_K : K \rightarrow \mathbb{R}$ be the restriction of Ψ_g to K . We recall that $\Psi_g|_K$ is said to be an equivariant Morse function if there are only finitely many critical orbits $\{\mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_k}\}$ in K , and the Hessian of $\Psi_g|_K$ is nondegenerate when restricted to the normal space $N_{p'}\mathcal{O}_{p_i}$ for any $p' \in \mathcal{O}_{p_i}$ and for any $p_i, i = 1, \dots, k$. We now define a subset \mathcal{G}_K of \mathcal{G} by collecting all g in \mathcal{G} such that the resulting potential function $\Psi_g|_K$ is an equivariant Morse function over K .

We are now ready to state main theorem of this chapter.

Theorem 3.1. *Let K be a $SO(n)$ -invariant, closed set in P consisting exclusively of nondegenerate configurations, then \mathcal{G}_K is open and dense in \mathcal{G} with respect to the Whitney C^k -topology, $1 \leq k \leq \infty$.*

The rest of this chapter is devoted to the proof of the main theorem, and is organized as follows. In section 2, we consider all possible partitions on the set of edges in Γ . We show that there is a partial order that reflects the granularity of the partition. The set of partitions, with this partial order, is then a graded lattice. In section 3, we use this graded lattice to generate a family of equivariant polynomial maps defined on the configuration space. The zero locus of each polynomial map is then a real semi-algebraic set, hence admits the canonical stratification, also known as the minimal semi-algebraic stratification. We then combine this family of canonical stratifications to generate a particular Whitney semi-algebraic stratification of the configuration space, as we call in this chapter, the Y -stratification of P . In section 4, we introduce the notion of perturbability of the gradient vector field, it is defined as a map from the configuration space to natural numbers. By doing this, we try to measure to what extent we can perturb the gradient vector field by varying function g over the configuration space. We show that the perturbability at each configuration at least matches the dimension of the stratum in the Y -stratification that contains the configuration. This fact, together with the analysis of the local structure of the Y -stratification as we will do in section 5, leads us to

the result of local genericity of equivariant Morse function, as we will prove in section 6. In section 7, we present a complete proof of the main theorem.

3.2 Graded lattice of edges

Let E be the set of edges in a complete graph on N vertices, i.e.,

$$E := \{(i, j) | 1 \leq i < j \leq N\} \quad (3.6)$$

A **partition** $\tau = (E_1, \dots, E_m)$ of E is a way to decompose E into a union of disjoint, nonempty subsets E_i , $i = 1, \dots, m$. Let \mathcal{T} be the collection of all partitions of E . There is a partial order on \mathcal{T} describing the granularity of the partition. Let $\tau = (E_1, \dots, E_m)$ and $\tau' = (E'_1, \dots, E'_{m'})$ be two partitions in \mathcal{T} , we say τ is **finer** than τ' or simply write $\tau \prec \tau'$ if $m > m'$ and for each E_j , there exists E'_k such that $E_j \subseteq E'_k$. The set \mathcal{T} , equipped with this partial order, is a bounded lattice. The greatest element is the trivial partition (E) while the least element is the edge-wise partition $((1, 2), (1, 3), \dots, (N-1, N))$.

We now review some useful notions about graded lattice. Suppose at this moment, \mathcal{T} is an arbitrary lattice. We say an lattice element τ a **cover** of τ' if $\tau \succ \tau'$ and there is no other lattice element τ'' such that $\tau \succ \tau'' \succ \tau'$. The lattice \mathcal{T} is said to be **graded**, or ranked, if it can be equipped with a rank function $\rho : \mathcal{T} \rightarrow \mathbb{N}$, compatible with the ordering, i.e., $\rho(\tau) < \rho(\tau')$ if $\tau \prec \tau'$, and whenever τ' covers τ , then $\rho(\tau') = \rho(\tau) + 1$. The value of the rank function for a lattice element is called its **rank**.

We now define, in our case, a graded structure on \mathcal{T} . A partition $\tau = (E_1, \dots, E_m)$ is said to be **elementary** if all but one E_k are singletons, and E_k contains exactly two edges. Given two partitions τ and τ' , we let $\tau \vee \tau'$ be their least upper bound, also known as the **join** of τ and τ' . Let \mathcal{T}_1 and \mathcal{T}_2 be two subset of \mathcal{T} , we then let

$$\mathcal{T}_1 \vee \mathcal{T}_2 := \{\tau_1 \vee \tau_2 | \tau_1 \in \mathcal{T}_1, \tau_2 \in \mathcal{T}_2\} \quad (3.7)$$

Let \mathcal{T}' be a subset of \mathcal{T} , we then define

$$\vee^k \mathcal{T}' := \underbrace{\mathcal{T}' \vee \cdots \vee \mathcal{T}'}_{k \text{ times}} \quad (3.8)$$

We now define the rank function $\rho : \mathcal{T} \rightarrow \mathbb{N}$.

Element of rank zero. Let τ^0 be the least element in \mathcal{T} , it acts as an identity under \vee operation, i.e, $\tau^0 \vee \tau = \tau$ for any $\tau \in \mathcal{T}$, and we define $\rho(\tau^0) := 0$.

Elements of rank one. Let \mathcal{T}^1 be the collection of all elementary partitions in \mathcal{T} , and we define $\rho(\tau) := 1$ for each $\tau \in \mathcal{T}^1$.

Elements of higher ranks. For each $k > 1$, we let

$$\mathcal{T}^k := \vee^k \mathcal{T}^1 - \vee^{k-1} \mathcal{T}^1 \quad (3.9)$$

and we define $\rho(\tau) := k$ for each $\tau \in \mathcal{T}^k$. In other words, if τ is generated by k elementary partitions under the \vee operation, and k is the least number τ can be generated in this way, then $\rho(\tau) = k$.

Since there are only finitely many elementary partitions, and in fact the cardinality of \mathcal{T}^1 is given by

$$|\mathcal{T}^1| = \frac{1}{8}(N+1)N(N-1)(N-2) \quad (3.10)$$

so all but finitely many \mathcal{T}^k are empty. We here establish an upper bound for k such that \mathcal{T}^k will be an empty set if k exceeds the upper bound.

Lemma 3.2.1. *Suppose $N \geq 2$, i.e, the number of agents is at least two, then \mathcal{T}^k is an empty set if $k \geq \frac{1}{2}N(N-1)$.*

Proof. The proof is done by induction on the number of agents.

Base case. Suppose $N = 2$, then \mathcal{T} is a singleton. So by definition $\mathcal{T}^1 = \emptyset$.

Inductive step. We assume that the lemma holds for $N = k$, $k \geq 2$, and we prove for the case

$N = k + 1$. Let E' be a subset of E defined by $E' := \{(i, j) | 1 \leq i < j \leq k\}$, and let

$$\tau' := (E', (1, k+1), \dots, (k, k+1)) \quad (3.11)$$

then by induction the rank of τ' is less than $\frac{1}{2}k(k-1)$. Let τ_i , $1 \leq i \leq k$, be an elementary partition defined by specifying the only non-singleton set as $\{(i, k+1), (i+1, k+1)\}$, then we have

$$(E) = \tau' \vee \tau_1 \vee \dots \vee \tau_k \quad (3.12)$$

so the rank of the greatest element in \mathcal{T} is less than $\frac{1}{2}(k+1)k$, and this establishes the lemma. \square

In this chapter, we will let R be the rank of the greatest element (E) in \mathcal{T} . So then \mathcal{T}^R is a singleton consisting only of the trivial partition $\tau^R := (E)$, and $\mathcal{T}^k = \emptyset$ for any $k > R$. Notice that each partition τ other than τ^0 can be generated by elementary partitions via the \vee operation, so we actually decompose \mathcal{T} into disjoint nonempty subsets with respect to the rank of partitions

$$\mathcal{T} = \bigcup_{k=0}^R \mathcal{T}^k \quad (3.13)$$

where \mathcal{T}^0 is a singleton consisting only of τ^0 . In other words, the lattice \mathcal{T} is a complete join-semilattice. In occasions, we will write τ^k when there is a need to emphasize the rank of a partition.

3.3 The Y -stratification of the configuration space

Let τ^1 be an elementary partition, and let $E_k = \{(i, j), (i', j')\}$ be the only non-singleton subset of E associated with the partition. We define an equivariant polynomial map $h_{\tau^1} : P \rightarrow \mathbb{R}$ by

$$h_{\tau^1}(p) := (|\vec{x}_i - \vec{x}_j|^2 - |\vec{x}_{i'} - \vec{x}_{j'}|^2)^2 \quad (3.14)$$

Let τ^k be a partition of rank k , and suppose $\tau^k = \vee_{i=1}^k \tau_i^1$, $k \geq 1$, we then define an equivariant polynomial map $h_{\tau^k} : P \rightarrow \mathbb{R}^k$ by

$$h_{\tau^k}(p) := (h_{\tau_1^1}(p), \dots, h_{\tau_k^1}(p)) \quad (3.15)$$

Let X_{τ^k} be a subset in P defined by

$$X_{\tau^k} := h_{\tau^k}^{-1}(0) \quad (3.16)$$

Notice that P is a semi-algebraic subset in $\mathbb{R}^{n \times N}$, verified by its definition, so X_{τ^k} is also a semi-algebraic subset. Here is a geometric interpretation of X_{τ^k} . Suppose $\tau = (E_1, \dots, E_m)$, then each configuration p in X_{τ^k} has to satisfy the condition that if two edges (i, j) and (i', j') are contained in the same E_l for some $l = 1, \dots, m$, then $d_{ij} = d_{i'j'}$. Notice that there may be multiple ways to express τ^k in terms of elementary partitions, but all of them will yield the same semi-algebraic set X_{τ^k} . In the case where $\tau = \tau^0$, we simply let $X_{\tau^0} := P$.

Notice that the partial order on \mathcal{T} induces a partial order on the family of semi-algebraic sets

$$X_{\mathcal{T}} := \{X_{\tau} | \tau \in \mathcal{T}\} \quad (3.17)$$

We observe that if $\tau \prec \tau'$, then $X_{\tau} \supseteq X_{\tau'}$. The join operation \vee in \mathcal{T} corresponds to the set intersection \cap in $X_{\mathcal{T}}$, i.e, $X_{\tau \vee \tau'} = X_{\tau} \cap X_{\tau'}$.

So $X_{\mathcal{T}}$ is also a graded lattice, and the “generators” are X_{τ^1} , $\tau^1 \in \mathcal{T}^1$. They will generate a semi-algebraic subset X_{τ} via intersection as long as $\tau \neq \tau^0$. But we also notice that the rank function ρ associated with \mathcal{T} may not be applied to $X_{\mathcal{T}}$ directly, i.e, the function $\rho' : X_{\mathcal{T}} \rightarrow \mathbb{R}$ defined by $\rho'(X_{\tau}) := \rho(\tau)$ may not be a rank function for the induced lattice $X_{\mathcal{T}}$. This is because there may exist $\tau \in \mathcal{T}$ with $\tau \prec \tau^R$ such that X_{τ} is an empty set, then $X_{\tau'}$ is an empty set for any $\tau' \succ \tau$, or there may exist a pair (τ, τ') with $\tau \succ \tau'$, but with $X_{\tau} = X_{\tau'}$. If we rule out empty sets from $X_{\mathcal{T}}$, then there may not exist the greatest element in $X_{\mathcal{T}}$.

We first recall some facts about stratification of real semi-algebraic sets. Let X be a semi-

algebraic subset in \mathbb{R}^m , a **semi-algebraic stratification** of X is a locally finite partition of X into smooth connected submanifolds of \mathbb{R}^m , the strata of X , such that each stratum is a semi-algebraic subset and the boundary of each stratum in \mathbb{R}^m is a union of lower dimensional strata. For $0 \leq j \leq m$, we define X^j to be the union of all strata of dimension j and refer to X^j as the **j -dimensional stratum** of X . In the rest of this chapter, we shall always regard a semi-algebraic stratification of X as consisting of the set of j -dimensional strata of X , $0 \leq j \leq m$, and we shall denote the semi-algebraic stratification of X by $\{X^j\}$. (see, for example, [19, 20], for works about stratifications).

A semi-algebraic stratification $\{X^j\}$ is said to be a **Whitney stratification** if for any two strata X^j and X^k , with $j < k$, the Whitney regularity condition holds at every point $x \in X^j \cap \overline{X^k}$:

Whitney regularity condition. If $\{a_i\}_{i \in \mathbb{N}}$ and $\{b_i\}_{i \in \mathbb{N}}$ are sequences of points of X^j and X^k respectively such that both sequences converge to x , the line joining a_i to b_i converges to a line L , and the tangent space $T_{b_i} X^k$ converges in the Grassmannian of k -dimensional subspace of \mathbb{R}^m to T , then $L \subset T$.

Whitney proved that each real algebraic set admits a Whitney stratification whose strata are real algebraic sets. By applying resolution of singularities, Hironaka proved that the same is true for real semi-algebraic set, with each stratum now a real semi-algebraic set.

A **minimal** semi-algebraic stratification is a Whitney stratification $\{X^j\}$ of X such that if $\{X'^j\}$ is any other semi-algebraic of X , then either the two stratifications are the same or there exists j_0 such that $X^j = X'^j$, $j > j_0$, and $X^{j_0} \supsetneq X'^{j_0}$. It is clear that if a minimal stratification exists, then it is unique. It is known that each real semi-algebraic set X admits the minimal semi-algebraic stratification (see [21]), often called the **canonical stratification** of X .

Back to our case, for each τ we let $\{X_\tau^j\}$ be the canonical stratification of X_τ . The minimality of stratification implies that each stratum is $SO(n)$ -invariant (see [20]). In other words, if X_τ^j contains p , then X_τ^j contains the whole orbit \mathcal{O}_p .

We will now use the family of the canonical stratifications $\{X_\tau^j\}$, $\tau \in \mathcal{T}$, to generate a special Whitney semi-algebraic stratification, as we call the **Y -stratification** of P and simply denote by $\{Y_k^j\}$. The strata of the Y -stratification are doubly indexed by the supindex j and the subindex k .

The supindex j of each stratum Y_k^j refers to the dimension of Y_k^j while the subindex k indicates its relation with \mathcal{T}^k , the set of rank k partitions.

Let Y_k be the union of all stratum Y_k^j over j , soon we will see that Y_k is a semi-algebraic set and $[Y_k^j]$ will be a Whitney semi-algebraic stratification of it. The Y -stratification of P will then be clear after we define for each $k = 1, \dots, L$, the subset $Y_k \subset P$ and its associated stratification $[Y_k^j]$. For convenience, we let

$$X_{\mathcal{T}^k}^j := \bigcup_{\tau \in \mathcal{T}^k} X_{\tau}^j \quad (3.18)$$

and let

$$X_{\mathcal{T}^k} = \bigcup_{j=0}^{\dim P} X_{\mathcal{T}^k}^j \quad (3.19)$$

The subset Y_k , together with its associated stratification $[Y_k^j]$ is defined below.

The semi-algebraic subset Y_R and its associated stratification. We recall that $\tau^R = (E)$ is the greatest element in \mathcal{T} . Let

$$Y_R := X_{\tau^R} \quad (3.20)$$

and for each $j = 1, \dots, \dim P$, we let

$$Y_R^j := X_{\tau^R}^j \quad (3.21)$$

So the stratification $[Y_R^j]$ of Y_R coincides with the canonical stratification $\{X_{\tau^R}^j\}$ of X_{τ^R} .

The semi-algebraic subset Y_k , $0 \leq k \leq R-1$, and its associated stratification. Let

$$Y_k := X_{\mathcal{T}^k} - X_{\mathcal{T}^{k+1}} \quad (3.22)$$

then Y_k is a semi-algebraic set because algebraic sets are closed under boolean operations. For each $j = 0, \dots, \dim P$, we let

$$Y_k^j := X_{\mathcal{T}^k}^j - X_{\mathcal{T}^{k+1}}^j \quad (3.23)$$

Then $[Y_k^j]$ is a Whitney semi-algebraic stratification of Y_k , and each stratum Y_k^j is a smooth subman-

ifold.

Notice that in the definition, we have abused the term by calling Y_k^j a stratum as each Y_k^j is actually a union of connected smooth semi-algebraic subsets. So to avoid confusion, in the rest of this chapter if there is a need to emphasize a connected stratum, then we will say it in an explicit way.

We here state the fact that the Y -stratification of P is a Whitney semi-algebraic stratification, i.e, it is a locally finite partition into smooth connected submanifolds each of which is a semi-algebraic subset, it satisfies a specific frontier condition as we will describe below, and it satisfies the Whitney regularity condition. We postpone the proof to the end of section 5, but in this chapter all we need is the frontier condition and the fact that each Y_k^j is a smooth submanifold.

The so called **frontier condition** of a stratification $\{X^j\}$ of a real semi-algebraic set $X \subset \mathbb{R}^m$ refers to the condition that the boundary of each j -dimensional stratum X^j in \mathbb{R}^m is contained in the union of lower dimensional strata. Yet, the situation here is a little different because the strata in the Y -stratification are doubly indexed by dimensions of the strata and ranks of partitions associated with the strata, so we have to specify the priority. We will now define a chain of strata in the Y -stratification in a way that the order of the chain is with respect to the frontier condition.

Lemma 3.3.1. *Let C be a chain of strata in the Y -stratification of P defined by*

$$C := Y_R^0 \rightarrow \cdots \rightarrow Y_R^{\dim P} \rightarrow Y_{R-1}^0 \rightarrow \cdots \rightarrow Y_{R-1}^{\dim P} \rightarrow \cdots \rightarrow Y_0^{\dim P} \quad (3.24)$$

The order of chain is with respect to the frontier condition, i.e, for each nonempty stratum Y_k^j in the chain, its boundary defined by $\overline{\partial Y_k^j} := \overline{Y_k^j} - Y_k^j$ (the closure is taken in P), is contained in the union of all strata before it, i.e,

$$\overline{\partial Y_k^j} \subseteq \bigcup_{i=0}^{j-1} Y_k^i \cup \bigcup_{i=k+1}^R Y_i \quad (3.25)$$

Proof. This is a consequence of the fact that each $X_{\mathcal{T}^{k+1}}$ is a subvariety of $X_{\mathcal{T}^k}$, so all sets $X_{\mathcal{T}^k}$ form a descending filtration as $P = X_{\mathcal{T}^0} \supseteq \cdots \supseteq X_{\mathcal{T}^R}$. \square

Remark. It is clear by equation (3.24) that the subindex k has priority over the supindex j with

respect to the frontier condition, and the subindex k is in a descending order while supindex j is in an ascending order.

We end this section with a discussion about the geometry of each stratum in the Y -stratification of P . Let $\tau = (E_1, \dots, E_m)$ be a partition, and let Y_τ be a semi-algebraic subset of P defined by the following condition: let p be a configuration in Y_τ , then $d_{ij} = d_{i'j'}$ if and only if two edges (i, j) and (i', j') are contained in the same E_l for some $l = 1, \dots, m$. In other words, if we define a **geometric pattern** of a configuration by specifying selected pairs of equal distances between agents, then Y_τ collects all configurations with the same geometric pattern defined by τ . The subset Y_k is then the union of Y_τ as τ varies over \mathcal{T}_k . From this point of view, we can also see that Y_k is $SO(n)$ -invariant because the geometric pattern is defined in a $SO(n)$ -invariant way as it only concerns relative distances between pairs of agents. Notice that there may be empty strata in the Y -stratification of P . So we may trim the chain defined in equation (3.24) by removing empty strata, and this modification won't affect the frontier condition.

3.4 Perturbability of the gradient vector field

Let p be a configuration in P , and let x_j^i be the i -th coordinate of agent \vec{x}_j . Let

$$\vec{x}_p^i := (x_1^i, \dots, x_N^i) \quad (3.26)$$

be a column vector in \mathbb{R}^N that consists of the i -th coordinates of agents in p . Rearrange entries of p by concatenating \vec{x}_p^i , $i = 1, \dots, n$, i.e.,

$$p = (\vec{x}_p^1; \dots; \vec{x}_p^n) \quad (3.27)$$

In the rest of this chapter, we will always assume that the entries of p are arranged in this way, and consequently we assume that entries of vectors in tangent space $T_p P$ and etc. are rearranged correspondingly.

Let M be a real N -by- N matrix, and we recall that M is said to be an **interaction matrix** if each column/row of M has zero sum. Let $G(p)$ be a symmetric interaction matrix defined by specifying its off-diagonal entries. Let $G_{ij}(p)$, $i \neq j$, be the ij -th entry of $G(p)$, and we define

$$G_{ij}(p) := g(d_{ij}) \quad (3.28)$$

where d_{ij} is the Euclidean distance between \vec{x}_i and \vec{x}_j in p . Let \mathcal{M} be the vector space of all N -by- N symmetric, interactions matrices, and let $G : P \rightarrow \mathcal{M}$ be defined by sending p to $G(p)$, then G can be thought of as a section of the trivial vector bundle $P \times \mathcal{M}$ over P .

With notations above, the gradient vector field $f_g(p)$ associated with the potential function Ψ_g can be written as

$$f_g(p) = \text{Diag}(G(p))p \quad (3.29)$$

where $\text{Diag}(\cdot)$ sends a N -by- N matrix M to a $(n \times N)$ -by- $(n \times N)$ block-diagonal matrix defined by

$$\text{Diag}(M) := \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix} \quad (3.30)$$

We will now investigate the perturbability of the gradient vector field f_g . Let p be a configuration in P , then there is a unique partition τ_p in \mathcal{T} such that Y_{τ_p} contains p . Suppose $\tau_p = (E_1, \dots, E_m)$, for each E_k , we define a symmetric interaction matrix A_k by specifying its off-diagonal entries. Let the ij -th, $i \neq j$, entry of A_k be one if $(i, j) \in E_k$ and be zero otherwise. We then let \mathcal{A}_p be a vector space of N -by- N interaction matrices spanned by $\{A_1, \dots, A_m\}$, and let F_p be a vector space in $\mathbb{R}^{n \times N}$ defined by

$$F_p := \{\text{Diag}(A)p \mid A \in \mathcal{A}_p\} \quad (3.31)$$

or equivalently, we can define

$$F_p = \{\text{Diag}(G(p))p \mid g \in \mathcal{G}\} \quad (3.32)$$

We now define the **perturbability** of the gradient vector field as a function $\xi : P \rightarrow \mathbb{N}$ by

$$\xi(p) := \dim F_p \quad (3.33)$$

The function ξ measures the perturbability of the gradient vector field $f_g(p)$ as p varies over the configuration space. Our goal in this section is to evaluate $\xi(p)$ by establishing an upper bound and a lower bound respectively.

3.4.1 An upper bound for $\xi(p)$

Our goal in this part is to establish an upper bound for $\xi(p)$, this is done by lemma 3.4.1.

Lemma 3.4.1. *Let $N_p\mathcal{O}_p$ be the normal space of the orbit \mathcal{O}_p at p , then $N_p\mathcal{O}_p$ contains F_p as a subspace.*

Proof. We first show that T_pP contains F_p as a subspace. Let \vec{e} be a vector of all ones in \mathbb{R}^N , then for each $A \in \mathcal{A}_p$ and for each $i = 1, \dots, n$, we have

$$\langle \vec{e}, A\vec{x}_p^i \rangle = 0 \quad (3.34)$$

where $\langle \cdot, \cdot \rangle$ is the normal inner-product of two vectors. So then $F_p \subset T_pP$.

Next we show $F_p \subset N_p\mathcal{O}_p$. Let $T_p\mathcal{O}_p$ be the tangent space of the orbit \mathcal{O}_p at p , then

$$T_p\mathcal{O}_p = \{(\Omega \otimes I_N)p | \Omega \in \mathfrak{so}(n)\} \quad (3.35)$$

where $\mathfrak{so}(n)$ is the set of n -by- n skew-symmetric matrices. Let $\vec{e}_1, \dots, \vec{e}_n$ be a standard basis for \mathbb{R}^n , and let

$$\Omega_{ij} := \vec{e}_i \vec{e}_j^T - \vec{e}_j \vec{e}_i^T \quad (3.36)$$

then the set of skew-symmetric matrices $\{\Omega_{ij} | 1 \leq i < j \leq n\}$ form a vector basis for $\mathfrak{so}(n)$. In fact, if p is nondegenerate, then $\{(\Omega_{ij} \otimes I_N)p | 1 \leq i < j \leq n\}$ is a vector basis for F_p . Let A be a matrix in

\mathcal{A}_p , so then $\text{Diag}(A)p$ is a vector in F_p . We have

$$\langle (\Omega_{ij} \otimes I_N)p, \text{Diag}(A)p \rangle = \langle -\vec{x}_p^j, A\vec{x}_p^i \rangle + \langle \vec{x}_p^i, A\vec{x}_p^j \rangle = 0 \quad (3.37)$$

This completes the proof. \square

Remark. Lemma 3.4.1 can also be proved by using equation (3.32). Each vector $\vec{v} \in F_p$ can be regarded as the restriction of certain gradient vector field at p . On the other hand, a gradient vector field always choose a direction along which the potential drops most quickly. So then $\vec{v} \in N_p\mathcal{O}_p$.

The function ξ changes its value as p varies over P . By lemma 3.4.1, we have $\xi(p) \leq \dim N_p\mathcal{O}_p$, and we will see later that, the equality $\xi(p) = \dim N_p\mathcal{O}_p$ holds for almost all configurations in P . Yet, there are exceptions, and here is an example.

Example. Consider a symmetric ring configuration in the plane, if we use complex coordinates, then $\vec{x}_p = (1, \omega, \dots, \omega^{N-1})$ with $\omega = \exp(-j\frac{2\pi}{N})$. Consequently, each matrix $A \in \mathcal{A}_p$ is circulant because of the symmetry of the ring configuration, so \vec{x}_p is an eigenvector of A with respect to a real eigenvalue. In other words, if we let \vec{v} and \vec{u} be the real part and the imaginary part of \vec{x}_p respectively, then $(A\vec{v}, A\vec{u}) = \lambda_A(\vec{v}, \vec{u})$. So then F_p is one dimensional vector space spanned by (\vec{v}, \vec{u}) and we have

$$\xi(p) = 1 < 2N - 3 = \dim N_p\mathcal{O}_p \quad (3.38)$$

So the perturbability is “degenerate” at the symmetric ring configuration.

The example above suggests that the perturbability is related to the geometric pattern. In fact, we observe that if there is a pair of equal distances $d_{ij} = d_{i'j'}$ in a configuration p , then $g(d_{ij})$ has to agree with $g(d_{i'j'})$. In other words, we have to perturb the ij -th entry and $i'j'$ -th entry of $G(p)$ simultaneously.

3.4.2 Infinitesimal rigidity of a nondegenerate configuration

We will show in this part that each nondegenerate configuration is infinitesimally rigid, this fact will be used later to establish a lower bound for $\xi(p)$. In graph theory, the **rigidity matrix** $R(\Gamma, p)$, with Γ a complete graph on N vertices at a configuration p , is given by

$$\begin{pmatrix} (\vec{x}_1 - \vec{x}_2)^T & (\vec{x}_2 - \vec{x}_1)^T & 0 & \cdots & 0 \\ (\vec{x}_1 - \vec{x}_3)^T & 0 & (\vec{x}_3 - \vec{x}_1)^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & (\vec{x}_{N-1} - \vec{x}_N)^T & (\vec{x}_N - \vec{x}_{N-1})^T \end{pmatrix} \quad (3.39)$$

The rank of $R(\Gamma, p)$ is at most $(n \times N) - \frac{1}{2}n(n+1)$ because the null space of $R(\Gamma, p)$ contains all infinitesimal rigid motions of p . If $R(\Gamma, p)$ achieves its full rank, then p is said to be **infinitesimally rigid**. (see, for example, [22] for works about rigidity matroids).

Theorem 3.4.2. *Let p be a nondegenerate configuration, then p is infinitesimally rigid.*

The proof of theorem 3.4.2 will be given after lemma 3.4.3. In lemma 3.4.3, we will consider a special case where we have $(n+1)$ agents in \mathbb{R}^n .

Lemma 3.4.3. *Suppose $N = n+1$, and p is a nondegenerate configuration, then p is infinitesimally rigid.*

Proof. We need to show in this case that the rank of $R(\Gamma, p)$ is $\frac{1}{2}n(n+1)$. The proof is done by induction on n .

Base case. Suppose $n = 1$, then the rigidity matrix is a row vector given by

$$R(\Gamma, p) = \left((\vec{x}_1 - \vec{x}_2)^T \quad (\vec{x}_2 - \vec{x}_1)^T \right) \quad (3.40)$$

Since $\vec{x}_1 \neq \vec{x}_2$, the rank of $R(\Gamma, p)$ is one.

Inductive step. Suppose the lemma holds for $n = k-1$, and we prove for the case $n = k$. Since P is

nondegenerate and $N = k + 1$, the rank of any sub-configuration of p formed by m agents is of rank m . So in particular, the subspace of least dimension in \mathbb{R}^k that embeds the sub-configuration formed by agents $\vec{x}_2, \dots, \vec{x}_k$ is a hyperplane. Let $R_1(\Gamma, p)$ be the restriction of $R(\Gamma, p)$ to the last $\frac{1}{2}k(k-1)$ row vectors, i.e, we let $R_1(\Gamma, p)$ be

$$\begin{pmatrix} 0 & (\vec{x}_2 - \vec{x}_3)^T & (\vec{x}_3 - \vec{x}_2)^T & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (\vec{x}_k - \vec{x}_{k+1})^T & (\vec{x}_{k+1} - \vec{x}_k)^T \end{pmatrix} \quad (3.41)$$

By induction, the matrix $R_1(\Gamma, p)$ achieves its full rank as $\frac{1}{2}k(k-1)$. Now let $R_2(\Gamma, p)$ be the restriction of $R(\Gamma, p)$ to the first k row vectors, i.e, we let $R_2(\Gamma, p)$ be

$$\begin{pmatrix} (\vec{x}_1 - \vec{x}_2)^T & (\vec{x}_2 - \vec{x}_1)^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\vec{x}_1 - \vec{x}_{k+1})^T & 0 & \cdots & (\vec{x}_{k+1} - \vec{x}_1)^T \end{pmatrix} \quad (3.42)$$

Since p is nondegenerate, the k vectors $\{\vec{x}_1 - \vec{x}_2, \dots, \vec{x}_1 - \vec{x}_{k+1}\}$ are then linearly independent. This implies that the rank of $R_2(\Gamma, p)$ is k , and the row vectors in $R_2(\Gamma, p)$ are linearly independent of the row vectors in $R_1(\Gamma, p)$. So then the matrix $R(\Gamma, p)$ achieves its full rank as $\frac{1}{2}k(k+1)$. \square

We now prove theorem 3.4.2.

Proof of theorem 3.4.2. We fix the dimension of the Euclidean space, and the proof is done by induction on the number of agents.

Base case. The case $N = n + 1$ is done by lemma 3.4.3.

Inductive step. We assume the lemma hold for $N = k$ with $k \geq n + 1$, and we prove for the case $N = k + 1$. We first state a fact proved in [23]. There exist $(n + 2)$ agents in p , say $\vec{x}_1, \dots, \vec{x}_{n+2}$, such that the two sub-configurations, formed by agents $\{\vec{x}_1, \dots, \vec{x}_{n+1}\}$ and by agents $\{\vec{x}_2, \dots, \vec{x}_{n+2}\}$ respectively, are both non-degenerate. We now assume this fact, and let $R_1(\Gamma, p)$ and $R_2(\Gamma, p)$ be the restrictions of $R(\Gamma, p)$ to the last $\frac{1}{2}k(k-1)$ row vectors and the first n row vectors respectively.

Then by induction, the matrix $R_1(\Gamma, p)$ achieves its full rank as $(k \times n) - \frac{1}{2}n(n+1)$ because the sub-configuration formed by $\vec{x}_2, \dots, \vec{x}_{k+1}$ is nondegenerate. On the other hand, the matrix $R_2(\Gamma, p)$ also achieves its full rank because the n vectors $\{\vec{x}_1 - \vec{x}_2, \dots, \vec{x}_1 - \vec{x}_{n+1}\}$ are linearly independent as the sub-configuration formed by $\vec{x}_1, \dots, \vec{x}_{n+1}$ is also non-degenerate. Moreover, the row vectors in $R_2(\Gamma, p)$ are linearly independent of the row vectors in $R_1(\Gamma, p)$. So the matrix $R(\Gamma, p)$ achieves its full rank as $((k+1) \times n) - \frac{1}{2}n(n+1)$. \square

3.4.3 A lower bound for $\xi(p)$

Our goal in this part is to establish a lower bound for $\xi(p)$, which we will state in theorem 3.4.4. We do this by relating $\xi(p)$ to the dimension of the stratum in the Y -stratification that contains p .

Theorem 3.4.4. *Let Y_k^j be a stratum of dimension j in the Y -stratification. Let p be a nondegenerate configuration in Y_k^j , then $\xi(p) \geq j - \dim \mathcal{O}_p$.*

Proof. For convenience, we let

$$j_p := j - \dim \mathcal{O}_p \quad (3.43)$$

If Y_k^j is nonempty, then $j_p \geq 0$ because each Y_k^j is $SO(n)$ -invariant. Let M be the connected stratum of Y_k^j that contains p , and we define a matrix map D over M by sending a configuration $p' \in M$ to the distance matrix $D(p')$ defined by

$$D(p') := \begin{pmatrix} -\sum_{i \neq 1} d_{1i}^2 & d_{12}^2 & \cdots & d_{1n}^2 \\ d_{12}^2 & -\sum_{i \neq 2} d_{i2}^2 & \cdots & d_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ d_{1n}^2 & d_{2n}^2 & \cdots & -\sum_{i \neq n} d_{in}^2 \end{pmatrix} \quad (3.44)$$

where d_{ij} is the Euclidean distance between \vec{x}_i and \vec{x}_j in p' . Let dD be the derivative of map D . Since all configurations in M share the same geometric pattern, so for each \vec{v} in the tangent space

$T_p M$, the matrix $dD(\vec{v})$ is contained in the vector space \mathcal{A}_p . Let

$$f_{\vec{v}} := \text{Diag}(dD(\vec{v}))p \quad (3.45)$$

then consequently $f_{\vec{v}}$ is contained in F_p .

Let $\{\vec{v}_1, \dots, \vec{v}_{j_p}\}$ be a set of independent vectors in $T_p M \cap N_p \mathcal{O}_p$, we show that the set of vectors $\{f_{\vec{v}_1}, \dots, f_{\vec{v}_{j_p}}\}$ in F_p are linearly independent. The proof is done by contradiction, i.e, we assume that there is a nonzero vector \vec{v} spanned by $\{\vec{v}_1, \dots, \vec{v}_{j_p}\}$ such that $f_{\vec{v}} = 0$.

To facilitate computation, we first define a family of N -by- N symmetric interaction matrices Δ^{ij} , $1 \leq i \leq j \leq n$, by specifying the off-diagonal entries of each Δ^{ij} . Let Δ_{st}^{ij} be the st -th, $s \neq t$, entry of Δ^{ij} , and we define

$$\Delta_{st}^{ij} := (x_s^i - x_t^i)(x_s^j - x_t^j) \quad (3.46)$$

here all coordinates are of agents in p . We then let Δ be a $(n \times N)$ -by- $(n \times N)$ symmetric, interaction matrix defined by

$$\Delta := \begin{pmatrix} \Delta^{11} & \Delta^{12} & \dots & \Delta^{1n} \\ \Delta^{12} & \Delta^{22} & \dots & \Delta^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{1n} & \Delta^{2n} & \dots & \Delta^{nn} \end{pmatrix} \quad (3.47)$$

With notations above, we then have

$$f_{\vec{v}} = \Delta \vec{v} \quad (3.48)$$

Next we define a vector \vec{u} by arranging back entries of \vec{v} . Write $\vec{v} = (\vec{v}^1; \dots; \vec{v}^n)$ with each \vec{v}^i a vector in \mathbb{R}^N , and write $\vec{u}^i = (u_1^i, \dots, u_N^i)$. For each $j = 1, \dots, N$, we let \vec{u}_j be a vector in \mathbb{R}^n defined by $\vec{u}_j := (u_j^1, \dots, u_j^n)$, and let \vec{u} be a vector in $\mathbb{R}^{n \times N}$ defined by $\vec{u} := (\vec{u}_1; \dots; \vec{u}_N)$. In other words, \vec{u} is defined from \vec{v} by reversing the rearrangement of entries we defined at the beginning of this section.

We then check that

$$\langle \vec{v}, f_{\vec{v}} \rangle = \langle \vec{v}, \Delta \vec{v} \rangle = - \sum_{1 \leq i < j \leq N} \langle \vec{x}_i - \vec{x}_j, \vec{u}_i - \vec{u}_j \rangle^2 = 0 \quad (3.49)$$

The equality holds if and only if \vec{u} lies in the null-space of the rigidity matrix $R(\Gamma, p)$ with Γ a complete graph on N vertices. The configuration p is nondegenerate, so by theorem 3.4.2, the matrix $R(\Gamma, p)$ achieves its full rank as $(n \times N) - \frac{1}{2}n(n+1)$. So the null-space of $R(\Gamma, p)$, after rearrangement of entries, is perpendicular to $N_p \mathcal{O}_p$. Since $v \in N_p \mathcal{O}_p$, the summation in equation (3.49) is strictly less than zero which contradicts to our assumption that $f_{\vec{v}} = 0$. \square

3.5 The local lattice and the local stratification

As \mathcal{T} is a bounded lattice, the induced lattice $X_{\mathcal{T}}$ ordered by inclusion is also bounded. The least element in $X_{\mathcal{T}}$ is $X_{\tau^0} = P$ while the greatest element in $X_{\mathcal{T}}$ is X_{τ^R} . However, as we mentioned earlier, there may be empty sets in $X_{\mathcal{T}}$. For example, consider a planar configuration of N agents, $N > 3$. The semi-algebraic set X_{τ^R} with respect to the greatest lattice element τ^R is empty because there is no planar configuration of N agents in P with all pairs of mutual distances equal to each other. In fact, X_{τ^R} is nonempty if and only if $N = n + 1$, and in any of such case, X_{τ^R} is diffeomorphic to $O(n)$, it consists of two orbits each of which is an orbit of a standard n -dimensional simplex. If we rule out all empty sets from $X_{\mathcal{T}}$, then the resulting collection of semi-algebraic sets may not be a bounded lattice anymore. In this section, we show that there is a complete sublattice \mathcal{T}_p of \mathcal{T} associated with each configuration p . The associated semi-algebraic subsets are nonempty, hence form a bounded lattice, and they together describe the local structure of the Y -stratification. We will also prove in this section that the Y -stratification of P is a Whitney semi-algebraic stratification.

3.5.1 The local lattice \mathcal{T}_p

Pick a configuration p , and we define the **local lattice**, as a subset of \mathcal{T} , to be

$$\mathcal{T}_p := \{\tau \in \mathcal{T} | p \in X_\tau\} \quad (3.50)$$

Before going on, we first define the notion of **complete sublattice**. Let τ and τ' be two elements in \mathcal{T} , we recall that the join of τ and τ' , denoted by $\tau \vee \tau'$, is defined to be the least upper bound of τ and τ' in \mathcal{T} . We now define the **meet** of τ and τ' , denoted by $\tau \wedge \tau'$, to be the greatest lower bound for τ and τ' in \mathcal{T} . Let \mathcal{T}' be a sublattice of \mathcal{T} , we say \mathcal{T}' is a complete sublattice if for any two elements τ and τ' in \mathcal{T}' , both the join and the meet of τ and τ' are contained in \mathcal{T}' .

Lemma 3.5.1. *Let p be a configuration, we recall that there is a unique partition τ_p such that $p \in Y_{\tau_p}$. Equivalently, we can define $\tau_p = (E_1, \dots, E_m)$ in a way that it satisfies the following condition. Let (i, j) and (i', j') be two edges in E , then two distances d_{ij} and $d_{i'j'}$ are equal to each other if and only if (i, j) and (i', j') are belong to the same subset E_k for some $k = 1, \dots, m$. Let \mathcal{T}_p be a sublattice defined by equation (3.50), then*

$$\mathcal{T}_p = \{\tau \in \mathcal{T} | \tau \preceq \tau_p\} \quad (3.51)$$

and \mathcal{T}_p is a complete sublattice. The least element in \mathcal{T}_p is τ^0 while the greatest element in \mathcal{T}_p is τ_p .

Proof. Notice that $X_{\tau^0} = P$, so each \mathcal{T}_p contains τ^0 as its least element. On the other side, by definition of τ_p , we see that $\tau_p \in \mathcal{T}_p$ and $\tau \preceq \tau_p$ for any $\tau \in \mathcal{T}_p$. So τ_p is the greatest element in \mathcal{T}_p . Let τ and τ' be two partitions in \mathcal{T}_p , we show both the join and the meet of τ and τ' are contained in \mathcal{T}_p .

Proof that $\tau \vee \tau' \in \mathcal{T}_p$. The configuration p is contained in both X_τ and $X_{\tau'}$. On the other hand, $X_{\tau \vee \tau'} = X_\tau \cap X_{\tau'}$, so p is also contained in $X_{\tau \vee \tau'}$, this then implies that $\tau \vee \tau'$ is contained in \mathcal{T}_p .

Proof that $\tau \wedge \tau' \in \mathcal{T}_p$. Since $\tau \wedge \tau' \preceq \tau_p$, we have $X_{\tau \wedge \tau'} \supseteq X_{\tau_p}$, so then $p \in X_{\tau \wedge \tau'}$, hence $\tau \wedge \tau' \in \mathcal{T}_p$. This argument also shows that if $\tau \preceq \tau_p$, then $\tau \in \mathcal{T}_p$. \square

The complete sublattice \mathcal{T}_p induces a graded structure from \mathcal{T} , i.e.,

$$\mathcal{T}_p := \bigcup_{k=0}^{R_p} \mathcal{T}_p^k \quad (3.52)$$

with $\mathcal{T}_p^k := \mathcal{T}_p \cap \mathcal{T}^k$. For each p , the set \mathcal{T}_p^0 is a singleton consisting of τ^0 while the number R_p is defined so that $\mathcal{T}_p^{R_p} = \{\tau_p\}$. Let

$$X_{\mathcal{T}_p} := \{X_\tau | \tau \in \mathcal{T}_p\} \quad (3.53)$$

and we call $X_{\mathcal{T}_p}$ the local family of X_τ at p . Similarly, the local family $X_{\mathcal{T}_p}$ has an induced lattice structure ordered by set inclusion, and moreover, each X_τ in the local family is nonempty.

The sublattice \mathcal{T}_p , by its definition, can be understood as the collection of all $\tau \in \mathcal{T}$ with X_τ intersecting p . Similarly, for any open neighborhood U of p , we can define a sublattice of \mathcal{T} by

$$\mathcal{T}_U := \{\tau | X_\tau \cap U \neq \emptyset\} \quad (3.54)$$

This definition apparently depends on the choice of U , yet \mathcal{T}_U will be stabilized after U is shrunk to a sufficiently small size.

Lemma 3.5.2. *There exists an open neighborhood U_p such that $\mathcal{T}_{U_p} = \mathcal{T}_p$. So for any open neighborhood U of p contained in U_p , we have $\mathcal{T}_U = \mathcal{T}_p$.*

Proof. Let τ be a partition in \mathcal{T} . There are two cases, either $\tau \in \mathcal{T}_p$, then $\tau \in \mathcal{T}_U$ for any open neighborhood U of p , or $\tau \notin \mathcal{T}_p$, then there exists an open neighborhood U_τ such that $U_\tau \cap X_\tau = \emptyset$. The set $\mathcal{T} - \mathcal{T}_p$ is finite, so if we let

$$U_p := \bigcap_{\tau \in \mathcal{T} - \mathcal{T}_p} U_\tau \quad (3.55)$$

then U_p is an open neighborhood of p , and by construction, U_p satisfies the condition in the statement of the lemma. \square

Lemma 3.5.2 somehow justifies the notion of the local lattice, and in fact we will use \mathcal{T}_p as the local data to pick a subset of strata in the Y -stratification that can be used to stratify any sufficiently

small open neighborhood of p .

3.5.2 The local stratification

We first recall the chain of strata we defined in equation (3.24), i.e.,

$$C = Y_R^0 \rightarrow \cdots \rightarrow Y_R^{\dim P} \rightarrow Y_{R-1}^0 \rightarrow \cdots \rightarrow Y_{R-1}^{\dim P} \rightarrow \cdots \rightarrow Y_0^{\dim P}$$

We now define the local chain C_p by removing any stratum Y_k^j from C if its closure $\overline{Y_k^j}$ in P does not contain p . Let n_p be the length of the local chain, and for convenience, we simply write

$$C_p := Y_p^1 \rightarrow \cdots \rightarrow Y_p^{n_p} \quad (3.56)$$

The supindex j of stratum Y_p^j does not indicate the dimension of Y_p^j anymore, but rather the ascending order of the supindices is with respect to the frontier condition of the local stratification as we will define now.

Let $B_\delta(p)$ be an open ball centered at p with radius δ . For each $j = 1, \dots, n_p$, we define

$$B_p^j := Y_p^j \cap B_\delta(p) \quad (3.57)$$

Our goal in this part is to show that $\{B_p^j\}$ is a Whitney semi-algebraic stratification of $B_\delta(p)$ if δ is sufficiently small. Notice that if this is the case, then we actually prove that the Y -stratification is a Whitney semi-algebraic stratification.

Theorem 3.5.3. *If δ is sufficiently small, then $\{B_p^j\}$ is a Whitney semi-algebraic stratification of $B_\delta(p)$.*

Proof. By lemma 3.5.2, we can choose a sufficiently small δ such that $B_\delta(p)$ is a union of B_p^j , i.e.,

$$B_\delta(p) = \bigcup_{j=1}^{n_p} B_p^j \quad (3.58)$$

By lemma 3.5.1, we have actually verified the frontier condition, i.e,

$$\overline{\partial B_p^j} \subseteq \bigcup_{i=1}^{j-1} B_p^i \quad (3.59)$$

So it suffices for us to show that for each $j = 1, \dots, n_p$, there are only finitely many connected strata in B_p^j each of which is a smooth semi-algebraic set, and the stratification $\{B_p^j\}$ satisfies the Whitney regularity condition.

We first show the finiteness of connected components of each B_p^j . It is clear that each B_p^j is a semi-algebraic subset because both Y_p^j and $B_\delta(p)$ are. We now define an auxiliary semi-algebraic subset by joining the configuration p to B_p^j , i.e, we define

$$\hat{B}_p^j := B_p^j \cup \{p\} \quad (3.60)$$

Consider the canonical stratification of \hat{B}_p^j , and we assume that the open ball $B_\delta(p)$ only intersects finitely many connected strata by shrinking radius δ . This then implies that there are finitely many connected strata in B_p^j as well, and each of which is a smooth semi-algebraic submanifold.

We now show that the stratification $\{B_p^j\}$ satisfies the Whitney regularity condition, i.e, we assume that $\{a_i\}_{i \in \mathbb{N}}$ and $\{b_i\}_{i \in \mathbb{N}}$ are two sequences of points of $B_p^{j_1}$ and $B_p^{j_2}$ respectively, both converging to a point $x \in B_p^{j_1} \cap \overline{B_p^{j_2}}$, and we assume further that the line joining a_i and b_i converges to a line L , and the tangent space $T_{b_i} B_p^{j_2}$ converges in the Grassmannian of $\dim B_p^{j_2}$ to T . We need to show that T contains L . By passing to subsequences if necessary, we may as well assume that there are $X_{\tau_1}^{j'_1}$ and $X_{\tau_2}^{j'_2}$ such that $\{a_i\}_{i \in \mathbb{N}} \subset X_{\tau_1}^{j'_1}$ and $\{b_i\}_{i \in \mathbb{N}} \subset X_{\tau_2}^{j'_2}$. Consequently $\tau_1 \succeq \tau_2$ because otherwise, the two sequences $\{a_i\}_{i \in \mathbb{N}}$ and $\{b_i\}_{i \in \mathbb{N}}$ won't converge to the same point. This then implies that $X_{\tau_1} \subseteq X_{\tau_2}$. So by passing to a subsequence of $\{a_i\}_{i \in \mathbb{N}}$, we may assume that $\{a_i\}_{i \in \mathbb{N}}$ is contained in a connected stratum of $X_{\tau_2}^{j'_1}$. But then, we have to conclude that $j'_1 \leq j'_2$. If $j'_1 = j'_2$, then the two sequences $\{a_i\}_{i \in \mathbb{N}}$ and $\{b_i\}_{i \in \mathbb{N}}$ are contained in the same connected submanifold, so naturally T contains L . If $j'_1 < j'_2$, we then use the fact that $\{X_{\tau_2}^j\}$ is a Whitney stratification. This then establishes the proof. \square

3.6 Local genericity on the space of nondegenerate configurations

In this section, we will investigate the local genericity of equivariant Morse functions. To be precise, we fix a nondegenerate configuration p , and show that there is a closed neighborhood W of p (understood as the closure of an open neighborhood of p in P) such that \mathcal{G}_W is open dense in \mathcal{G} .

3.6.1 Theorem 3.6.1

We recall the open ball $B_\delta(p)$ is defined so that $\{B_p^j\}$ is a Whitney semi-algebraic stratification. The closed neighborhood W of p we will work with is assumed to be contained in $B_\delta(p)$. To facilitate analysis, we further impose two conditions on W , and they are

Condition I.1. The closed neighborhood W is compact, and consists only of nondegenerate configurations. This can be done by shrinking W .

Condition I.2. The closed neighborhood W is $SO(n)$ -invariant, this can be done by letting $SO(n)$ act on W .

Notice that the $SO(n)$ -action on W neither violate the condition that W only intersects strata in the local chain C_p , nor does it conflict with condition I.1 because both the strata and the space of nondegenerate configurations are $SO(n)$ -invariant. In the rest of this chapter, we shall always assume that the closed neighborhood W satisfies these two conditions. Our goal in this section is to establish theorem 3.6.1.

Theorem 3.6.1. *The subset \mathcal{G}_W is open and dense in \mathcal{G} with respect to the Whitney C^k -topology, $1 \leq k \leq \infty$.*

The proof of theorem 3.6.1 relates to the local stratification. Before going on, we note some works related to the analysis of the genericity assumption. In [24], M. J. Field establishes the Kupka-Smale density theorem for equivariant dynamical systems. In [25], R. Pignoni proved the density and stability of Morse functions on a stratified space. Yet, the local genericity we will establish in this

section is not an direct application of these results. Because in our case, the choices we can make for perturbing the gradient vector field are restricted. So along the proof, we will establish a technique that can be generalized to investigate other genericity properties on a stratified space. The proof has three key ingredients, they are the frontier condition, the fact that p is infinitesimally rigid and the lower bound for the perturbability. We will now exploit these three ingredients in the next three parts.

3.6.2 Use of the frontier condition

We start with a definition of a technical condition. That is

Condition T.1. Let Y_k^j be any stratum in the Y -stratification of P , and let Z be any $SO(n)$ -invariant, compact subset of $Y_k^j \cap W$. Then \mathcal{G}_Z is open and dense in \mathcal{G} .

We will verify this condition later in this section, and our goal in this part is to assume *condition T.1* and prove theorem 3.6.1.

Theorem 3.6.2. *If condition T.1 holds, then \mathcal{G}_W is open and dense in \mathcal{G} with respect to the Whitney C^k -topology.*

Proof. We recall that the local chain C_p defined in equation (3.56) is given by

$$C_p := Y_p^1 \rightarrow \cdots \rightarrow Y_p^{n_p}$$

Let

$$W^j := Y_p^j \cap W \tag{3.61}$$

Then $\{W^j\}$ is a stratification of W . The proof of genericity of \mathcal{G}_W proceeds in a way that is with respect to the frontier condition.

Base case. The stratum W^0 is a closed subset in W by the frontier condition, and W is chosen to be compact, so W^0 is also a compact set. By *condition T.1*, we conclude that \mathcal{G}_{W^0} is open and dense in \mathcal{G} . Perturb g if necessary so that after perturbation, the function g lies in \mathcal{G}_{W^0} . Since W^0 is compact, there exists a $SO(n)$ -invariant, open neighborhood U_0 of W^0 in P and an open neighborhood V_0 of g

in \mathcal{G} such that \mathcal{G}_{W^0} contains V_0 as a subset.

Inductive step. Let

$$\widehat{W}^k := \bigcup_{i=0}^k W^i \quad (3.62)$$

We assume at this moment that there is a $SO(n)$ -invariant, open neighborhood U_k of \widehat{W}^k in P and an open neighborhood V_k of g in \mathcal{G} such that \mathcal{G}_{U_k} contains V_k as a subset. We now proceed one step further to prove that the same will hold for \widehat{W}^{k+1} . Let

$$Z_{k+1} := W^{k+1} - U_k \quad (3.63)$$

then Z_{k+1} is $SO(n)$ -invariant, and by the frontier condition, it is a compact subset of W^{k+1} . By *condition T.1*, we conclude that $\mathcal{G}_{Z_{k+1}}$ is open and dense in \mathcal{G} . Perturb g if necessary so that after perturbation, the function g lies in $\mathcal{G}_{Z_{k+1}} \cap V_k$. So then,

$$g \in \mathcal{G}_{Z_{k+1}} \cap V_k \subset \mathcal{G}_{Z_{k+1}} \cap \mathcal{G}_{U_k} \subset \mathcal{G}_{\widehat{W}^{k+1}} \quad (3.64)$$

Since \widehat{W}^{k+1} is compact, there is a $SO(n)$ -invariant, open neighborhood U_{k+1} of \widehat{W}^{k+1} in P and an open neighborhood V_{k+1} of g in \mathcal{G} such that $\mathcal{G}_{U_{k+1}}$ contains V_{k+1} as a subset. This then completes the inductive step.

The induction terminates at the step $k = n_p$ as by definition $\widehat{W}^{n_p} = W$. So we have succeeded in perturbing g so that $g \in \mathcal{G}_W$, and meanwhile finding an open neighborhood V_{n_p} of g such that \mathcal{G}_W contains V_{n_p} as a subset. In other words, \mathcal{G}_W is open and dense in \mathcal{G} . \square

The rest of this section is then to verify *condition T.1*.

3.6.3 Use of the fact that p is infinitesimally rigid

We recall the $(n \times N)$ -by- $(n \times N)$ symmetric, interaction matrix defined by equation (3.47). Previously we have showed that if p is nondegenerate, then p is infinitesimally rigid, and hence Δ is

nondegenerate restricted to the normal space $N_p\mathcal{O}_p$. Our goal in this part is to establish lemma 3.6.3.

Lemma 3.6.3. *Let p be a nondegenerate configuration, and let Δ be defined by equation (3.47). Let H be a $(n \times N)$ -by- $(n \times N)$ symmetric, interaction matrix and its null space contains the tangent space $T_p\mathcal{O}_p$ as a subspace. Let Z be a subset of \mathbb{R} defined by collecting all z such that the matrix $H + z\Delta$ is nondegenerate when restricted to the normal space $N_p\mathcal{O}_p$. Then Z is open and dense in \mathbb{R} with respect to the Euclidean topology.*

Proof. Openness of Z is clear, and we show the density. Suppose $H + z\Delta$ is degenerate when restricted to $N_p\mathcal{O}_p$, we show there is an ϵ_0 such that if $0 < |\epsilon| < \epsilon_0$, then $H + (z + \epsilon)\Delta$ is nondegenerate when restricted to $N_p\mathcal{O}_p$. For convenience, we let

$$L := n \times N - \frac{1}{2}n(n+1) \quad (3.65)$$

be the dimension of the normal space $N_p\mathcal{O}_p$. Let $\vec{v}_1, \dots, \vec{v}_L$ be orthonormal eigenvectors of $H + z\Delta$ that are contained in $N_p\mathcal{O}_p$, and let $\lambda_1, \dots, \lambda_L$ be the corresponding eigenvalues. Without loss of generality, we may assume that $\lambda_1 = \dots = \lambda_k = 0$ while the rest are nonzero. There is an $\epsilon_0 > 0$ such that if we perturb z to $z + \epsilon$ with $|\epsilon| < \epsilon_0$, then each nonzero eigenvalue remains nonzero after perturbation. Let Δ' be a k -by- k symmetric matrix defined by

$$\Delta' := (v_1, \dots, v_k)^T \Delta (v_1, \dots, v_k) \quad (3.66)$$

Then Δ' is nonsingular because Δ is nondegenerate when restricted to $N_p\mathcal{O}_p$. Let $\lambda'_1, \dots, \lambda'_k$ be eigenvalues of Δ' , and let $\tilde{\lambda}_i$, $1 \leq i \leq L$, be eigenvalues of $H + (z + \epsilon)\Delta$. Then by an appropriate reordering if necessary, we have for each $i = 1, \dots, k$

$$\tilde{\lambda}_i = \epsilon \lambda'_i + o(\epsilon^2) \quad (3.67)$$

In other words, $\epsilon \lambda'_i$, $1 \leq i \leq k$, are the first order approximation of perturbed zero eigenvalues. We may shrink ϵ_0 if necessary, so then $H + (z + \epsilon)\Delta$ is nondegenerate when restricted to $N_p\mathcal{O}_p$ for any ϵ

with $0 < |\epsilon| < \epsilon_0$. □

3.6.4 Use of the lower bound for $\xi(p)$

Let p be a nondegenerate configuration in a stratum Y_k^j , we have showed in theorem 3.4.4 that $\dim F_p \geq j_p$ with $j_p = j - \dim \mathcal{O}_p$. So we can choose a set of vectors $\{f_{v_1}, \dots, f_{v_{j_p}}\}$ in F_p such that

$$\langle (f_{v_k} - f_g(p)), (f_{v_l} - f_g(p)) \rangle = \delta_{kl} \quad (3.68)$$

where δ_{kl} is the Kronecker delta. Choose a set of functions $\{g_1, \dots, g_{j_p}\} \subset \mathcal{G}$ correspondingly so that each g_k satisfies the condition $f_{g_k}(p) = f_{v_k}$, this can be done because

$$F_p = \{ \text{Diag}(G(p))p | g \in \mathcal{G} \} \quad (3.69)$$

Notice that by choosing the family of functions $\{g_1, \dots, g_{j_p}\}$, we then have

$$\langle (f_{g_k}(\theta \cdot p) - f_g(\theta \cdot p)), (f_{g_l}(\theta \cdot p) - f_g(\theta \cdot p)) \rangle = \delta_{kl} \quad (3.70)$$

for any $\theta \in SO(n)$. Our goal in this part is prove theorem 3.6.4.

Theorem 3.6.4. *Let p be a nondegenerate configuration in a stratum Y_k^j of the Y -stratification. Choose a set of functions $\{g_1, \dots, g_{j_p}\}$ such that equation (3.70) holds. Let S be a $SO(n)$ -invariant, closed neighborhood of p in Y_k^j , and we assume that $f_{g_1}(p'), \dots, f_{g_{j_p}}(p')$ are linearly independent for any $p' \in S$. We further assume that S is a smooth manifold, possibly with boundary, consisting exclusively of nondegenerate configurations. Then \mathcal{G}_S is open and dense in \mathcal{G} .*

Proof. The openness of \mathcal{G}_S is clear by compactness of S , so we here prove the density of \mathcal{G}_S in \mathcal{G} .

For each $p' \in S$, we let $Q(p')$ be a j_p -by- j_p matrix defined by taking the first j_p rows of the matrix formed by the column vectors $f_{g_1}(p'), \dots, f_{g_{j_p}}(p')$. Without loss of generality, we assume that $Q(p')$ is nonsingular for any $p' \in S$. Correspondingly, we let $q_g(p')$ be a truncated vector of $f_g(p')$ by

taking its first j_p entries. Define a map

$$\chi_g : S \rightarrow \mathbb{R}^{j_p} \quad (3.71)$$

by sending p' to $Q^{-1}(p')q_g(p')$. Then by Sard's theorem, for almost all vectors $\vec{u} = (u_1, \dots, u_{j_p})$, the map $\chi_g + \vec{u}$ is regular at each of its critical points. We now relate the perturbation of g to the perturbation of χ_g . Let

$$\tilde{g} := g + \sum_{k=1}^{j_p} u_k g_k \quad (3.72)$$

then

$$\chi_{\tilde{g}} = \chi_g + \vec{u} \quad (3.73)$$

So we can perturb g if necessary such that χ_g is a regular map.

As the dimension of S is j , so then the dimension of each connected critical manifold associated with χ_g is $j - j_p = \dim \mathcal{O}_p$. Since S is compact, there are only finitely many connected critical manifolds (possibly with boundaries) in S , and we label them as $\{M_1, \dots, M_n\}$. We now relate this set of critical manifolds to the set of equilibria orbits associated with Ψ_g . Let $p' \in S$ be an equilibrium associated with the potential function Ψ_g , then for each $p'' \in \mathcal{O}_{p'}$, we have $q_g(p'') = 0$ because $q_g(p'')$ is a truncation of $f_g(p'')$. In other words, $\mathcal{O}_{p'}$ is a critical manifold associated with χ_g . So if $p' \in M_i$, then M_i has to coincide with $\mathcal{O}_{p'}$ because both M_i and $\mathcal{O}_{p'}$ are connected and $\dim M_i = \dim \mathcal{O}_{p'}$. Conversely, if $p' \in M_i$ is not an equilibrium associated with Ψ_g , then neither is any other configuration in M_i . In other words, the collection of equilibria orbits associated with Ψ_g in S , labeled as $\{\mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_l}\}$, is a subset of $\{M_1, \dots, M_n\}$.

The compactness of S also implies that there is an open neighborhood V of g in \mathcal{G} such that for any $\tilde{g} \in V$, the map $\chi_{\tilde{g}}$ is regular. Let $\{\tilde{M}_1, \dots, \tilde{M}_{\tilde{n}}\}$ be the set of perturbed critical manifolds, then we may shrink V if necessary such that the total number of critical manifolds remains to be the same, i.e, $\tilde{n} = n$, and if M_i isn't an equilibria orbit, then neither is \tilde{M}_i . This can be done because each critical manifold M_i is actually a continuous function of $g \in V$.

Let p_k , $1 \leq k \leq l$, be an equilibrium in S . The Hessian of Ψ_g at each p_k can be derived by

computing the the derivative of the gradient vector field f_g at p_k , and is given by

$$df_g(p_k) = \text{Diag}(G(p_k)) + \begin{pmatrix} H^{11}(p_k) & \cdots & H^{1n}(p_k) \\ \vdots & \ddots & \vdots \\ H^{1n}(p_k) & \cdots & H^{nn}(p_k) \end{pmatrix} \quad (3.74)$$

Each $H^{ab}(p_k)$, $1 \leq a \leq b \leq n$, is a N -by- N symmetric, interaction matrix. Let H_{st}^{ab} be the st -th, $s \neq t$, entry of $H^{ab}(p)$ defined by

$$H_{ab,st}(p_k) := \frac{g'(d_{st})}{d_{st}}(x_s^a - x_t^a)(x_s^b - x_t^b) \quad (3.75)$$

all coordinates here are of agents in p_k . We will now show that by perturbing g if necessary, each Hessian matrix $df_g(p_k)$, $1 \leq k \leq l$, is nondegenerate when restricted to $N_{p_k} \mathcal{O}_{p_k}$.

Fix an equilibrium, say p_1 , and we let Δ be the $(n \times N)$ -by- $(n \times N)$ interaction matrix defined by equation (3.47), with each x_j^i now the i -th coordinate of agent \vec{x}_j in p_1 . Now compare Δ and $df_g(p_1)$, we then find that if we let $g(d_{ij}) = 0$ for all distances d_{ij} in p_1 , and let $g'(d_{ij}) = d_{ij}$, then $df_g(p) = \Delta$. This, in particular, implies that if we perturb g by fixing values of $g(d_{ij})$ for all mutual distances d_{ij} in p_1 , while perturbing the derivative $g'(d_{ij})$ to be $\tilde{g}'(d_{ij}) := g'(d_{ij}) + \epsilon d_{ij}$, then the perturbed Hessian of $\Psi_{\tilde{g}}$ at p_1 will be

$$df_{\tilde{g}}(p_1) = df_g(p_1) + \epsilon \Delta \quad (3.76)$$

By lemma 3.6.3, for almost all $\epsilon \in \mathbb{R}$, the matrix $df_g(p_1) + \epsilon \Delta$ is nondegenerate when restricted to $N_{p_1} \mathcal{O}_{p_1}$. So we can perturb g inside V such that after perturbation, each \mathcal{O}_{p_k} , $k = 1, \dots, m$, will still be a critical orbit and the Hessian $df_g(p_k)$ will be nondegenerate when restricted to $N_{p_k} \mathcal{O}_{p_k}$. Notice that the Hessian of Ψ_g will then be nondegenerate when restricted to $N_{p'} \mathcal{O}_{p_k}$ for any $p' \in \mathcal{O}_{p_k}$ and for any $k = 1, \dots, m$. This is because the set of eigenvalues of the Hessian is invariant along a critical orbit. So at this moment, we have successfully perturbed g so that after perturbation g is contained in \mathcal{G}_S . \square

Remark (Remark 1). Suppose χ_g is a regular map and suppose $g \in \mathcal{G}_S$. Then if we perturb g , each equilibria orbit \mathcal{O}_{p_k} will remain in Y_k^j after perturbation because the map χ_g is regular at \mathcal{O}_{p_k} .

Remark (Remark 2). In the proof, we assume that the vector space F_p takes its least possible dimension j_p . Now suppose $\dim F_p > j_p$. then correspondingly, $Q(p')$ is a $\dim F_p$ -by- $\dim F_p$ matrix and $q(p')$ is a truncated vector of dimension $\dim F_p$. Now suppose $\chi_g : S \rightarrow \mathbb{R}^{\dim F_p}$ is a regular map, and let M be a connected critical manifold associated with χ_g , then

$$\dim M = j - \dim F_p < \dim \mathcal{O}_p \quad (3.77)$$

Notice that if $j - \dim F_p < 0$, then M is an empty set. The inequality above then implies that there is no equilibria orbit associated with Ψ_g in S .

3.6.5 Proof of theorem 3.6.1

To prove theorem 3.6.1, it suffices for us to verify *condition T.1*, and it is done by corollary 3.6.5.

Corollary 3.6.5. *Let Y_k^j be a stratum in the Y -stratification of P , and let Z be a $SO(n)$ -invariant, compact subset of $Y_k^j \cap W$, then \mathcal{G}_Z is open and dense in \mathcal{G} .*

Proof. Openness of \mathcal{G}_Z is clear by compactness of Z , and we prove the density of \mathcal{G}_Z . Pick a configuration p' in Z , and let $S_{p'}$ be a $SO(n)$ -invariant, closed neighborhood of p' in Y_k^j such that $\mathcal{G}_{S_{p'}}$ is open and dense in \mathcal{G} . Then the family of $S_{p'}$ as p' varies over Z is a cover for Z . Since Z is compact, there is a finite cover of Z by $\{S_{p_1}, \dots, S_{p_n}\}$ as a subset of $\{S_{p'} | p' \in Z\}$. It is clear that

$$\bigcap_{i=1}^n \mathcal{G}_{S_{p_i}} \subseteq \mathcal{G}_Z \quad (3.78)$$

Since a finite intersection of open and dense sets is still open and dense, so then \mathcal{G}_Z is dense as it contains an open and dense subset. \square

Theorem 3.6.1 is then proved by combining theorem 3.6.2 and corollary 3.6.5.

3.7 Proof of main theorem

Our goal in this section is to prove theorem 3.1. Before going on, we first recall the main theorem we proved in chapter I.

Compact global attractor. Suppose the interaction function g satisfies conditions of non-vanishing attraction and non-vanishing repulsion, then the set of equilibria associated with Ψ_g is compact. In particular, there exist two numbers l_- and l_+ with $0 < l_- < l_+ < \infty$ such that the distance between any two agents in an equilibrium lies in the closed interval $[l_-, l_+]$. Moreover, the gradient flow of each initial condition exists for all time and converges to the set of the equilibria.

We will now use this fact to prove theorem 3.1.

Proof of theorem 3.1. Let l_- and l_+ be defined above, and let $a := l_-/2$ and $b := 2l_+$. Let P_a^b be a $SO(n)$ -invariant subset of P defined by

$$P_a^b := \{p \in P \mid a \leq d_{ij} \leq b, \forall i < j\} \quad (3.79)$$

and let $K_a^b := K \cap P_a^b$. As an intersection of two compact sets, the subset K_a^b is also compact. Let p be a configuration in K_a^b , let W_p be a $SO(n)$ -invariant, closed neighborhood of \mathcal{O}_p in P such that \mathcal{G}_{W_p} is open and dense in \mathcal{G} . The family of W_p as p varies over K_a^b is an open cover of K_a^b . Since K_a^b is compact, there is a finite set $\{p_1, \dots, p_m\} \subset K_a^b$ such that $\{W_{p_1}, \dots, W_{p_m}\}$ is a finite cover of K_a^b . Notice that

$$\mathcal{G}_{K_a^b} \subseteq \bigcap_{k=1}^m \mathcal{G}_{W_{p_k}} \quad (3.80)$$

So $\mathcal{G}_{K_a^b}$ is open and dense in \mathcal{G} .

Since P_a^b is compact, there is an open neighborhood V of g in \mathcal{G} such that the set of equilibria associated with $\Psi_{\tilde{g}}$ is contained in P_a^b for any $\tilde{g} \in V$. Perturb g , if necessary, so that after perturbation g is contained in $\mathcal{G}_{K_a^b} \cap V$. On the other hand, the set \mathcal{G}_K contains $\mathcal{G}_{K_a^b} \cap V$ as a subset, so then g is contained in \mathcal{G}_K . Moreover, if V_g is an open neighborhood of g contained in $\mathcal{G}_{K_a^b}$, then \mathcal{G}_K contains $V_g \cap V$ as an open subset. In other words, the set \mathcal{G}_K is open and dense in \mathcal{G} . \square

Chapter 4

Formation Control with Controllable Interactions

4.1 Definitions and main theorems

Over the last two decades, formation control has been one of the most-studied topics in multi-agent systems. A general question about formation control is to ask how interconnected agents communicate and work together in a centralized/decentralized fashion towards common goals. This question has been investigated in different models with various applications (see, for example, [2, 6, 9, 11, 26–32]).

In this chapter, we consider a control model as a variation of the class of reciprocal multi-agent (RMA) systems. The control of the formation is achieved by varying the magnitude of selected interactions between agents. Let $\Gamma = (V, E)$ be an undirected graph, with $V := \{1, \dots, N\}$ the set of vertices and E the set of edges, that describes the pattern of interaction. Let $V(i) := \{j \in V | ij \in E\}$ be the set of vertices that are adjacent to i . As a variation of a RMA system, the equations of motion, for the N agents $\vec{x}_1, \dots, \vec{x}_N \in \mathbb{R}^n$ with $N > n$, take the form

$$\dot{\vec{x}}_i = \sum_{j \in V(i)} u_{ij}(\vec{x}_j - \vec{x}_i), \quad i = 1, \dots, N \quad (4.1)$$

Each u_{ij} , $ij \in E$, controls the interaction between \vec{x}_i and \vec{x}_j . We require $u_{ij} = u_{ji}$ for all $ij \in E$, in other words, interactions between agents are reciprocal.

We will investigate this control model from two perspectives. One is from the view of centralized control, we will show that if there is a centralized controller that knows the position of each agent, and takes control of each u_{ij} , $ij \in E$, then system (4.1) is approximately path controllable. The other is from the view of decentralized control, we will consider a specific case with a particular pattern of information flow. A decentralized control law that relates to a gradient flow will be investigated in detail.

In this chapter, we will still restrict ourselves to configurations with zero centroid, yet we allow the situation that multiple agents are located at the same spot as we take into account of problems about consensus of agents, or the so called rendezvous problem. So the configuration space P is

defined by

$$P := \{(\vec{x}_1; \dots; \vec{x}_N) \in \mathbb{R}^{n \times N} \mid \sum_{i=1}^N \vec{x}_i = 0\} \quad (4.2)$$

Notice that there is a shape space related to the configuration space, it is defined as the quotient of the configuration space by ruling out the effects of rotation. The shape space often matters in situations where Euclidean coordinates of agents are irrelevant. However, in our case we do emphasize the Euclidean embedding of each configuration, i.e, two configurations of the same shape are considered to be different if they are embedded in different ways.

Before stating the main theorems, we make a few definitions. First we recall that a configuration p in P is said to be **nondegenerate** if it satisfies two conditions

1. $\vec{x}_i \neq \vec{x}_j$ for all $i \neq j$.
2. $\vec{x}_1 - \vec{x}_2, \dots, \vec{x}_1 - \vec{x}_N$ span \mathbb{R}^n .

Otherwise, we say p is **degenerate**. We remind the readers a geometric fact that p is nondegenerate if and only if there is no collision of agents and p can't be embedded into a lower dimensional Euclidean space in \mathbb{R}^n . In this chapter, we will let P_n and P_d be the space of nondegenerate configurations and the space of degenerate configurations, respectively.

Let Q be a subset in P . We say Q is **path-connected** if for any two configurations $p_0, p_1 \in Q$, there is a continuous function $\gamma : [0, 1] \rightarrow P$ with $\gamma(0) = p_0$ and $\gamma(1) = p_1$ such that the entire image of γ lies in Q .

We now state below the main theorems of this chapter.

Theorem 4.1. *Let N be the number of agents, and let n be the dimension of the Euclidean space of agents. If $n > 1$ and $N - n > 1$, then P_n , the space of nondegenerate configurations, is path-connected. Assume this condition, and we consider system (4.1) as a centralized control system. If the network topology Γ is connected, then system (4.1) is approximately path-controllable over P_n . The approximate path-controllability is defined as follows. Let $T > 0$ be a time period and let $\gamma : [0, T] \rightarrow P_n$ be a smooth curve in P_n , then there is a control law for each u_{ij} , $ij \in E$, such that by applying the law, we will be*

able to steer the system along a trajectory $\hat{\gamma}$ with $\hat{\gamma}(0) = \gamma(0)$, and $\hat{\gamma}$ can be made arbitrarily close to γ , i.e., $|\hat{\gamma}(t) - \gamma(t)| < \epsilon$ for any $t \in [0, T]$ for a given positive number ϵ .

Theorem 4.2. Consider system (4.1) as a decentralized control system. Let $\Gamma = (V, E)$ be a minimally connected graph with the set of edges defined by

$$E := \{1i | i = 2, \dots, N\} \quad (4.3)$$

Let $\hat{p} = (\hat{x}_1; \dots; \hat{x}_N)$ be a target configuration in P . Suppose each agent \vec{x}_i with $i \neq 1$,

1. has the coordinates of himself and the coordinates of agent \vec{x}_1 ;
2. knows the position of \hat{x}_1 and the position of \hat{x}_i ;
3. takes control of u_{1i} .

Then there is a decentralized control rule such that if each agent \vec{x}_i , $i \neq 1$, obeys the rule, then the control model will be a gradient-like system, minimizing the potential function defined by

$$\Psi(\vec{x}_1, \dots, \vec{x}_N) := \sum_{i=1}^N |\vec{x}_i - \hat{x}_i|^2 \quad (4.4)$$

and the only stable equilibrium associated with the system is \hat{p} .

The rest of chapter is organized as follows. In section 4.2, we recall the notion of interaction matrix, and compute the matrix Lie algebra associated with it. The result will then be used in section 4.3 to establish approximate path-controllability. In section 4.4, we investigate a concrete example with a particular decentralized information flow, and we show there is an artificial potential function that can be used to provide a decentralized solution for organizing multi-agent systems.

4.2 Matrix Lie algebra of interaction matrices

Let M be a N -by- N matrix, we recall M is said to be an **interaction matrix** if each column and each row of M has zero sum. Let (i, j) be a pair of vertices with $1 \leq i < j \leq N$, we define a symmetric N -by- N interaction matrix as

$$A_{ij} := \vec{e}_i \vec{e}_j^T + \vec{e}_j \vec{e}_i^T - \vec{e}_i \vec{e}_i^T - \vec{e}_j \vec{e}_j^T \quad (4.5)$$

with each \vec{e}_i is a standard basis element in \mathbb{R}^N .

Let $\Gamma = (V, E)$ be a connected, undirected graph. Let $A_\Gamma := \{A_{ij} | ij \in E\}$ be the set of interaction matrices associated with Γ . We then let \mathfrak{g} be the matrix Lie algebra generated by A_Γ . We will see soon that the Lie algebra \mathfrak{g} doesn't depend on Γ as long as the graph is connected.

Theorem 4.2.1. *If the graph Γ is connected, then the matrix Lie algebra \mathfrak{g} generated by matrices in A_Γ consists of all interaction matrices.*

The proof of theorem 4.2.1 will be given after lemma 4.2.2. We first recall some facts about graded Lie algebra. A Lie algebra \mathfrak{g} is \mathbb{Z}_2 -**graded** if it is a disjoint union of two parts as $\mathfrak{g} = \mathfrak{g}_0 \cup \mathfrak{g}_1$ such that

$$\begin{aligned} [\mathfrak{g}_0, \mathfrak{g}_0] &\subseteq \mathfrak{g}_0 \\ [\mathfrak{g}_1, \mathfrak{g}_1] &\subseteq \mathfrak{g}_0 \\ [\mathfrak{g}_0, \mathfrak{g}_1] &\subseteq \mathfrak{g}_1 \end{aligned} \quad (4.6)$$

In our case, if we let \mathfrak{g}_0 and \mathfrak{g}_1 be the two subsets of \mathfrak{g} consisting of skew-symmetric matrices and symmetric matrices, respectively, then it defines a $\mathbb{Z}/2$ grading on \mathfrak{g} . The proof of theorem 4.2.1 will be done if we can show that \mathfrak{g}_0 is the set of all skew-symmetric interaction matrices while \mathfrak{g}_1 is the set of all symmetric interaction matrices.

Let $\alpha := \{(i, j) | 1 \leq i < j \leq N\}$ be the collection of pairs of vertices, and we define a set of interaction matrices as

$$A := \{A_{ij} | ij \in \alpha\} \quad (4.7)$$

There are $\frac{1}{2}N(N-1)$ matrices in A , and they span the vector space of all symmetric N -by- N interaction matrices.

Example. We consider the case $N = 4$, then the set A consists of six symmetric, interaction matrices, and they are

$$\begin{aligned} A_{12} &= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & A_{13} &= \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & A_{14} &= \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \\ A_{23} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & A_{24} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} & A_{34} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \end{aligned}$$

These six matrices form a basis for the space of 4-by-4 symmetric, interaction matrices.

Let $\beta := \{(i, j, k) | 1 \leq i < j < k \leq N\}$ be the collection of triplets of vertices, and for each triplet $(i, j, k) \in \beta$, we define a skew-symmetric N -by- N interaction matrix as

$$B_{ijk} := (\vec{e}_i \vec{e}_k^T - \vec{e}_k \vec{e}_i^T) - (\vec{e}_i \vec{e}_j^T - \vec{e}_j \vec{e}_i^T) - (\vec{e}_j \vec{e}_k^T - \vec{e}_k \vec{e}_j^T) \quad (4.8)$$

Let B' be the collection of B_{ijk} as (i, j, k) varies over β , yet the matrices in B' are, in general, linearly dependent. So instead, we define

$$B := \{B_{1jk} | 1 < j < k \leq N\} \quad (4.9)$$

There are $\frac{1}{2}(N-1)(N-2)$ matrices in B and they span the vector space of all skew-symmetric N -by- N interaction matrices.

Example. We consider the case $N = 4$, then the set B consists of three skew-symmetric, interaction

matrices, and they are

$$B_{123} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_{124} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \quad B_{134} = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

These three matrices form a basis for the space of 4-by-4 skew-symmetric, interaction matrices.

Lemma 4.2.2. *The set of N -by- N interaction matrices is closed under Lie bracket, and*

$$\begin{aligned} [A_{ij}, A_{jk}] &= [A_{jk}, A_{ik}] = [A_{ik}, A_{ij}] = B_{ijk} \\ [A_{ij}, B_{ijk}] &= 2(A_{jk} - A_{ik}) \\ [A_{jk}, B_{ijk}] &= 2(A_{ik} - A_{ij}) \\ [A_{ik}, B_{ijk}] &= 2(A_{ij} - A_{ik}) \end{aligned} \tag{4.10}$$

Proof. Let \vec{e} be a vector in \mathbb{R}^N that consists of all ones. Let A and B be two interaction matrices, then

$$[A, B]\vec{e} = AB\vec{e} - BA\vec{e} = 0 \tag{4.11}$$

$$\vec{e}^T[A, B] = \vec{e}^T AB - \vec{e}^T BA = 0 \tag{4.12}$$

The rest directly follows the computation □

We now prove theorem 4.2.1.

Proof of theorem 4.2.1. The proof is done by induction on the number of vertices.

base case. There is nothing to prove if $N = 1$ because \mathfrak{g} is empty in this case.

induction step. We assume that the theorem holds for $N \leq m - 1$, and we prove for the case $N = m$.

Pick a chain of subgraphs

$$\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m = \Gamma \tag{4.13}$$

such that each Γ_k is a connected subgraph consisting exactly of k vertices. This can be done because Γ is connected. Let V_k and E_k be the set of vertices and the set of edges in Γ_k , respectively. Relabel the vertices, if necessary, so that $V_k = \{1, \dots, k\}$ for each $k = 1, \dots, m$.

The subgraph Γ_{m-1} is connected, so by induction, the following two sets of matrices are both contained in \mathfrak{g} :

$$A_{m-1} := \{A_{ij} | 1 \leq i < j \leq m-1\} \quad (4.14)$$

$$B_{m-1} := \{B_{1jk} | 1 < j < k \leq m-1\} \quad (4.15)$$

Since the graph Γ is connected, there exists an edge im for some i with $i < m$. We may assume $i = 1$, but then by lemma 4.2.2, the matrix A_{1m} together with A_{1i} , $1 < i < m$ will generate B_{1im} and A_{im} , $1 < i < m$. \square

Though the matrix Lie algebra \mathfrak{g} doesn't depend on Γ as long as the graph is connected, the generating process certainly does. We say \mathfrak{g} is **k -th bracket generatable** if it is spanned by $A_\Gamma, [A_\Gamma, A_\Gamma], \dots, [A_\Gamma, \dots, [A_\Gamma, A_\Gamma]]$ up to the k -th bracket and k is the least number that the spanning condition holds.

Lemma 4.2.3. *Let Γ be a connected graph, and let $d(\Gamma)$ be the diameter of Γ . Suppose \mathfrak{g} is k -th generatable, then $k \leq 2\lceil \log_2 d(\Gamma) \rceil + 1$ where $\lceil a \rceil$ denotes the least integer that bounds a real number a from above.*

Proof. Let j and k be a pair of distinct vertices in Γ , and we assume that the distance between j and k in Γ is $l \leq d(\Gamma)$. We choose a chain of vertices i_1, \dots, i_l with $i_1 = j$ and $i_l = k$ such that i_m and i_{m+1} are adjacent in Γ for all $m = 1, \dots, l-1$. We may relabel the vertices, if necessary, so that $i_1 < \dots < i_l$. We make our first claim that the matrix A_{jk} can be derived in no more than $2\lceil \log_2 l \rceil$ steps of matrix Lie bracketing. By lemma 4.2.2, we know that if the two matrices $A_{i_a i_b}$ and $A_{i_b i_c}$, $1 \leq a < b < c \leq l$, are available, then $A_{i_a i_c}$ can be derived in no more than two steps of matrix Lie bracketing. We then consider the following question: let $\Gamma' = (V', E')$ be a subgraph of Γ with $V' := \{i_1, \dots, i_l\}$ and $E' := \{i_m i_{m+1} | 1 \leq m \leq l-1\}$, then how many steps we need to take

to generate a new edge for Γ' by following the rule that an edge $i_a i_c$ can be generated if and only if both $i_a i_b$ and $i_b i_c$ are available? The answer to this question is $\lceil \log_2 l \rceil$, this then establishes the first claim. We now make our second claim that each matrix B_{ijk} , $1 \leq i < j < k \leq N$, can be derived in less than $2\lceil \log_2 d(\Gamma) \rceil + 1$ steps of matrix Lie bracketing. By lemma 4.2.2, if two matrices A_{ij} and A_{jk} are available, then B_{ijk} will be derived in one step of matrix Lie bracketing, but we have just showed that both A_{ij} and A_{jk} can be derived in no more than $2\lceil \log_2 d(\Gamma) \rceil$ steps of matrix Lie bracketing. This then establishes the second claim, and we complete the proof. \square

Example. We consider three examples illustrated in figure 12.

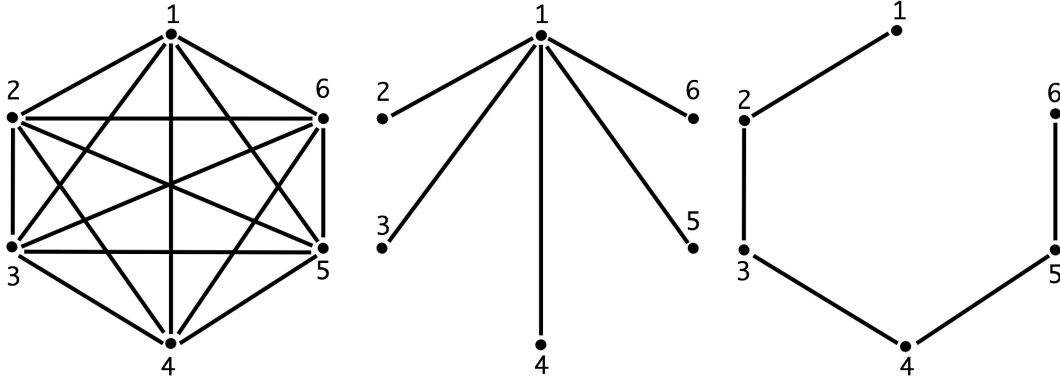


Figure 12: Three connected graphs on 6 vertices.

The graph on the left is a complete graph and has diameter 1, and the matrix Lie algebra \mathfrak{g} associated is 1-st bracket generatable. The two graphs in the middle and on the right are both minimally connected. The diameter of the middle graph in the figure is 2, and the matrix Lie algebra \mathfrak{g} is 2-nd bracket generatable. The diameter of the right graph in the figure is 5, and the matrix Lie algebra \mathfrak{g} is 7-th bracket generatable.

4.3 Approximate path-controllability on the space of nondegenerate configurations

Let p be a configuration in P , and let $\vec{x}_p^i = (x_1^i, \dots, x_N^i)$ be a vector in \mathbb{R}^N that consists of the i -th coordinates of agents. Rearrange entries of p so that

$$p := (\vec{x}_p^1, \dots, \vec{x}_p^n) \quad (4.16)$$

We will assume this arrangement through the rest of this section.

Let M be a N -by- N matrix and for convenience, let $\langle M \rangle$ be a $(n \times N)$ -by- $(n \times N)$ block-diagonal matrix as:

$$\langle M \rangle := \text{Diag}(M, \dots, M) \quad (4.17)$$

Let A_{ij} be the interaction matrix defined by equation (4.5), and let $f_{ij}(p) := \langle A_{ij} \rangle p$, then the control model described by equation (4.1) takes the standard form of an affine-control system

$$\dot{p} = \sum_{ij \in E} u_{ij} f_{ij}(p) \quad (4.18)$$

Our goal in this section is to prove theorem 4.1. We first discuss about the path-connectivity of P_n .

Lemma 4.3.1. *Let N be the number of agents, and let n be the dimension of the Euclidean space of agents. If $n > 1$ and $N > n + 1$, then P_n , the space of nondegenerate configurations, is path-connected.*

Proof. For each pair ij with $1 \leq i < j \leq N$, we let $P_{ij} := \{p \in P \mid \vec{x}_i = \vec{x}_j\}$. The codimension of each P_{ij} is n in P . Let $P_{\mathbb{R}^{n-1}}$ be the space of configurations that can be embedded in $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ with the last coordinate zero, and let P'_d be the space of configurations that can be embedded into lower dimensional Euclidean spaces, then

$$P'_d \approx \mathbb{RP}^{n-1} \times P_{\mathbb{R}^{n-1}} \quad (4.19)$$

where \mathbb{RP}^{n-1} is the real projective space parametrizing all linear subspaces of codimension one in \mathbb{R}^n .

So then, the dimension of P'_d is $(n-1)N$ and hence, the codimension of P'_d is $(N-n)$ in P . On the other hand, we have

$$P_n = P - \bigcup_{i < j} P_{ij} - P'_d \quad (4.20)$$

the codimension of each excluded subset is greater than one, so P_n is path-connected. \square

Remark (I). The space of nondegenerate configurations is disconnected if $n = 1$, and there are as many as $N!$ path-connected components. Assume that all agents are aligned on the x -axis, we then characterize each component by the ascending order of the x -coordinates of agents: Let S_N be the group of permutations on vertices $\{1, \dots, N\}$, and suppose $\sigma \in S_N$, then a component associated with σ is given by $\{p \in P | x_{\sigma(1)} < \dots < x_{\sigma(N)}\}$.

Remark (II). The space of nondegenerate configurations is disconnected if $N = n + 1$. In this case, there are exactly two path-connected components in P_n . Each component is characterized by the sign of the determinant of the n -by- n matrix $(\vec{x}_1 - \vec{x}_2, \dots, \vec{x}_1 - \vec{x}_{n+1})$.

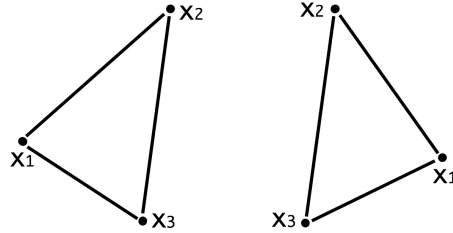


Figure 13: If $N = n + 1$, then P_n is disconnected. For example, we consider triangles in the plane. Each path in P that connects the left triangle in the figure to the right one has to intersect at least one line formation.

Remark (III). The quotient space P_n/S_N is always connected even if $n = 1$ or $N = n + 1$. If labels of agents are irrelevant, then the underlying space should be considered as P/S_N rather than P .

The tangent space $T_p P$ at each configuration p can be identified with P , but to avoid confusion, we use T to denote the tangent space. We now show that the control system described by equation (4.18) satisfies the Lie algebra rank condition, i.e, we show

Theorem 4.3.2. *Let \mathcal{L} be the Lie algebra spanned by vector fields f_{ij} as ij varies over E , let \mathcal{L}_p be the vector space spanned by $f(p)$ as f varies over \mathcal{L} . If the configuration p is nondegenerate, then $\mathcal{L}_p = T$.*

The proof of theorem 4.3.2 will be given after lemma 4.3.3 and lemma 4.3.4. We start by working on a special case where $N = n + 1$.

Lemma 4.3.3. *Let N be the number of agents, and let n be the dimension of the Euclidean space of agents. Suppose $N = n + 1$, then $\mathcal{L}_p = T$ for any nondegenerate configuration p .*

Proof. Let G_1 and G_2 be two interaction matrices in \mathfrak{g} . Notice that the operator $\langle \cdot \rangle$ commutes with the Lie bracket:

$$[\langle G_1 \rangle, \langle G_2 \rangle] = \langle [G_1, G_2] \rangle \quad (4.21)$$

so the Lie algebra \mathcal{L}_p is given by

$$\mathcal{L}_p = \{ \langle G \rangle p \mid G \in \mathfrak{g} \} \quad (4.22)$$

Recall that the union of two sets $A \cup B$ is a basis for the set of interaction matrices. If we can show that the set $\{ \langle G \rangle p \mid G \in A \cup B \}$ is a vector basis for \mathcal{L}_p , then $\dim \mathcal{L}_p = \dim T = (N - 1)^2$ and hence $\mathcal{L}_p = T$.

Let X_p be a N -by- N matrix defined by

$$X_p := (\vec{x}_p^1, \dots, \vec{x}_p^n, \vec{e}) \quad (4.23)$$

with $\vec{e} \in \mathbb{R}^N$ a vector of all ones. If there is a matrix $G \in \mathfrak{g}$ such that $\langle G \rangle p = 0$, then equivalently $GX_p = 0$. Since p is nondegenerate, X_p is then nonsingular, so the equality holds if and only if $G = 0$. \square

We now make a useful observation:

Lemma 4.3.4. *Let N be the number of agents, and let n be the dimension of the Euclidean space of agents, and we assume here $N > n + 1$. Let p be a nondegenerate configuration, then there are $(n + 2)$*

agents $\{\vec{x}_{i_1}, \dots, \vec{x}_{i_{n+2}}\}$ among p such that the two sub-configurations formed by agents $\{\vec{x}_{i_1}, \dots, \vec{x}_{i_{n+1}}\}$ and by agents $\{\vec{x}_{i_1}, \dots, \vec{x}_{i_n}, \vec{x}_{i_{n+2}}\}$ are both nondegenerate.

Proof. Since p is nondegenerate, there exist $(n+1)$ agents, say $\vec{x}_1, \dots, \vec{x}_{n+1}$, such that they form a nondegenerate sub-configuration \hat{p}_1 . Pick an agent, say \vec{x}_k , out of $\{\vec{x}_{n+2}, \dots, \vec{x}_N\}$, and if we can pick n agents out of $\{\vec{x}_1, \dots, \vec{x}_{n+1}\}$ such that they, together with \vec{x}_k , form another nondegenerate sub-configuration \hat{p}_2 , then we are done.

Suppose not, then for each $i = 1, \dots, n+1$, we let H_i be the hyperplane in \mathbb{R}^n spanned by $\{\vec{x}_1, \dots, \vec{x}_{i-1}, \vec{x}_{i+1}, \dots, \vec{x}_{n+1}\}$ with \vec{x}_i excluded. Let $K_j := \bigcap_{i \neq j} H_i$ be the transversal intersection of hyperplanes, then K_j is a singleton and consists only of \vec{x}_j . On the other hand, the agent \vec{x}_k has to be contained in each hyperplane H_i because otherwise, the sub-configuration formed by agents $\{\vec{x}_1, \dots, \vec{x}_{i-1}, \vec{x}_{i+1}, \dots, \vec{x}_{n+1}, \vec{x}_k\}$ is non-degenerate. So then we have to conclude that $\vec{x}_k = \vec{x}_i$ for all $i = 1, \dots, n+1$ which is a contradiction. \square

We are now ready to prove theorem 4.3.2.

Proof of theorem 4.3.2. Fix the dimension of the Euclidean space, the proof is then done by induction on the number of agents.

Base case. The case $N = n+1$ is done by lemma 4.3.3.

Induction step. Suppose the theorem holds for $N \leq m-1$, we prove for the case $N = m$ with $m > n+1$. By lemma 4.3.4 we may assume that the two sub-configurations formed by $\{\vec{x}_1, \dots, \vec{x}_{n+1}\}$ and by $\{\vec{x}_1, \dots, \vec{x}_n, \vec{x}_m\}$ are nondegenerate. Let A_{m-1} and B_{m-1} be two subsets of \mathfrak{g} define by equation (4.14) and (4.15), respectively. Since the sub-configuration formed by agents $\{\vec{x}_1, \dots, \vec{x}_{m-1}\}$ is nondegenerate, so by induction, the set $V' := \{\langle G \rangle p | G \in A_{m-1} \cup B_{m-1}\}$ contains $n(m-2)$ linearly independent vectors in \mathcal{L}_p .

On the other hand, the dimension of the tangent space is $n(m-1)$, so it suffices to find a subset V'' of \mathcal{L}_p such that V'' contains n linearly independent vectors in \mathcal{L}_p , and they are all linearly independent of vectors in V' .

Let \vec{v} be a vector in \mathcal{L}_p , write $\vec{v} = (\vec{v}^1; \dots; \vec{v}^n)$ with each $\vec{v}^i \in \mathbb{R}^m$, and for each \vec{v}^i , we write $\vec{v}^i = (v_1^i, \dots, v_m^i)$. If $\vec{v} \in V'$, then $v_m^i = 0$ for each $i = 1, \dots, n$, this is because the last row/column of $G \in A_m \cup B_m$ is zero. Let $\vec{v}_m := (v_m^1, \dots, v_m^n)$, then any vector $\vec{v} \in \mathcal{L}_p$ with $\vec{v}_m \neq 0$ will be linearly independent of V' .

Our choice of V'' is associated with the sub-configuration formed by agents $\{\vec{x}_1, \dots, \vec{x}_n, \vec{x}_m\}$. Let

$$V'' := \{\langle A_{im} \rangle p \mid 1 \leq i \leq n\} \quad (4.24)$$

and let $V_m'' := \{\vec{v}_m \mid \vec{v} \in V''\}$, then

$$V_m'' = \{\vec{x}_m - \vec{x}_1, \dots, \vec{x}_m - \vec{x}_n\} \quad (4.25)$$

It suffices to show that vectors in V_m'' are linearly independent, but this is true because the sub-configuration formed by agents $\{\vec{x}_1, \dots, \vec{x}_n, \vec{x}_m\}$ is nondegenerate. \square

We are now ready to prove theorem 4.1.

Proof of theorem 4.1. By lemma 4.3.1, the space of nondegenerate configurations is path-connected. By theorem 4.3.2, the control system satisfies the Lie algebra rank condition for system vector fields f_{ij} , $ij \in E$. So by Sussmann and Liu [33], we may frequently oscillate the controls u_{ij} to generate the Lie brackets of the vector fields f_{ij} , $ij \in E$. And by doing this, we can steer the system to approximate any smooth curve in P_n . \square

Remark. The space of nondegenerate configurations is open and dense in the configuration space. A degenerate configuration is easily perturbed to be nondegenerate. So practically, the control model, with a centralized controller, is approximately path-controllable on the whole configuration space.

4.4 Formation control using artificial potential functions

Graph theory is a natural tool for modeling patterns of information exchange among agents. Digraphs are particularly important in modeling network characteristics such as “agent i receives information from agent j , or “agent i measures its distance from agent j ”.

In our case, there is an important characteristics associated with the network: a directed graph $\Sigma = (V, A)$ that describes the pattern of information flow. A directed edge $i \rightarrow j$ in A refers to “agent i perceives agent j and takes control of u_{ij} ”. Notice that the digraph Σ is a directed version of Γ , i.e, if $i \rightarrow j$ is an edge in Σ , then ij is an edge in Γ .

So far we have implicitly assumed that there is a centralized controller that coordinates all the agents in the control system. In such case, a digraph Σ is irrelevant because each agent obeys the rule sent by the centralized controller. In this section, we will work on a specific graph Γ with two different patters of information flow. One is a realization of centralized formation control while the other is decentralized.

4.4.1 Centralized formation control

Let $\Gamma = (V, E)$ be a minimally connected graph defined by

$$E := \{1i | 2 \leq i \leq N\} \quad (4.26)$$

and let $\Sigma = (V, A)$ be defined by

$$A := \{1 \rightarrow i | 2 \leq i \leq N\} \quad (4.27)$$

In other words, the agent \bar{x}_1 perceives and coordinates all the other agents. Theorem 4.1 then implies that the system is approximately path-controllable.

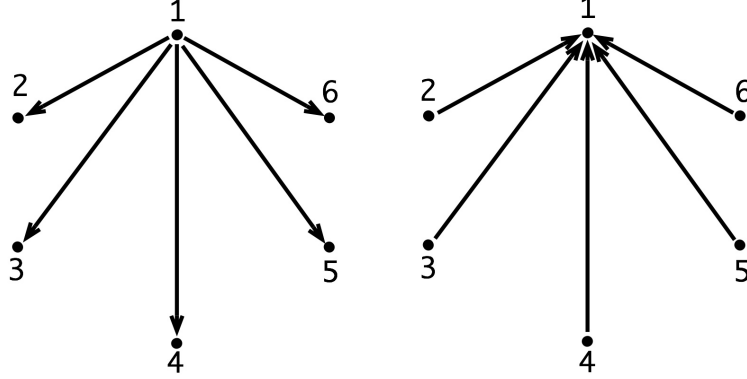


Figure 14: The digraph on the left side corresponds to a model of centralized formation control. The agent \vec{x}_1 acts as a leader and coordinates all the other agents. The digraph on the right side corresponds to a model of decentralized formation control. Each agent \vec{x}_i , $i \neq 1$, has the coordinates of himself and \vec{x}_1 , and it take control of u_{1i} . While \vec{x}_1 , like a puppet, perceives nothing but himself.

4.4.2 Decentralized formation control

We assume the same interaction pattern Γ , but reverse the direction of each edge in Σ , i.e,

$$A := \{i \rightarrow 1 | 2 \leq i \leq N\} \quad (4.28)$$

Our goal here is to prove theorem 4.2. As the decentralized control law is related to a gradient flow, so the proof will be clear after we define the potential function and establish certain properties of it. We start by working on a two-agents system:

Lemma 4.4.1. *Consider a control system of two agents \vec{x}_1, \vec{x}_2 in \mathbb{R}^n . The equations of motion of \vec{x}_1 and \vec{x}_2 are:*

$$\dot{\vec{x}}_1 = -\dot{\vec{x}}_2 = \tilde{u} \cdot \vec{e}_{12} \quad (4.29)$$

with

$$\vec{e}_{12} := \begin{cases} (\vec{x}_2 - \vec{x}_1)/|\vec{x}_2 - \vec{x}_1| & \vec{x}_1 \neq \vec{x}_2 \\ 0 & \vec{x}_1 = \vec{x}_2 \end{cases} \quad (4.30)$$

and \tilde{u} is the control. Let $\hat{\vec{x}}_1$ and $\hat{\vec{x}}_2$ be two fixed points in \mathbb{R}^n , and let $\Psi_{12} : \mathbb{R}^{2 \times n} \rightarrow \mathbb{R}$ be a smooth

function defined by

$$\Psi_{12}(\vec{x}_1, \vec{x}_2) := \frac{1}{2}(|\vec{x}_1 - \hat{\vec{x}}_1|^2 + |\vec{x}_2 - \hat{\vec{x}}_2|^2) \quad (4.31)$$

If we let

$$\tilde{u} := -\langle \vec{e}_{12}, (\vec{x}_1 - \hat{\vec{x}}_1) - (\vec{x}_2 - \hat{\vec{x}}_2) \rangle \quad (4.32)$$

and let $\vec{x}_1(t)$, $\vec{x}_2(t)$ be solutions of equation (4.29) at time t , then $\frac{d}{dt}\Psi_{12}(\vec{x}_1(t), \vec{x}_2(t)) = -u(t)^2$.

We omit the proof as lemma 4.4.1 directly follows the computation. We now prove theorem 4.2.

Proof of theorem 4.2. Define a potential function $\Psi : P \rightarrow \mathbb{R}$ by

$$\Psi(\vec{x}_1, \dots, \vec{x}_N) := \frac{1}{2} \sum_{i=1}^N |\vec{x}_i - \hat{\vec{x}}_i|^2 \quad (4.33)$$

and define a control law for each agent \vec{x}_i , $i \neq 1$, by

$$\dot{\vec{x}}_i = \langle \vec{e}_{1i}, (\vec{x}_1 - \hat{\vec{x}}_1) - (\vec{x}_i - \hat{\vec{x}}_i) \rangle \cdot \vec{e}_{1i} \quad (4.34)$$

then consequently we have

$$\dot{\vec{x}}_1 = - \sum_{i=2}^N \langle \vec{e}_{1i}, (\vec{x}_1 - \hat{\vec{x}}_1) - (\vec{x}_i - \hat{\vec{x}}_i) \rangle \cdot \vec{e}_{1i} \quad (4.35)$$

Let $\vec{x}_i(t)$, $i = 1, \dots, N$, be solutions of equation (4.34) and equation (4.35), then

$$\frac{d}{dt}\Psi(\vec{x}_1(t), \dots, \vec{x}_N(t)) = - \sum_{i=2}^N |\dot{\vec{x}}_i(t)|^2 \quad (4.36)$$

We now show that there is only one stable equilibrium \hat{p} associated with this control law. First notice that \hat{p} is a stable equilibrium because it is the global minima. Suppose $p' = (\vec{x}'_1; \dots; \vec{x}'_N)$ is

another equilibrium. Let $\Delta \vec{x}_i := \hat{x}_i - \vec{x}'_i$ and define $\Delta p := (\Delta \vec{x}_1; \dots; \Delta \vec{x}_N)$. Notice that

$$\sum_{i=1}^N \Delta \vec{x}_i = \sum_{i=1}^N \hat{x}_i - \sum_{i=1}^N \vec{x}'_i = 0 \quad (4.37)$$

so $p' + \epsilon \Delta p$, the perturbation of p' with $\epsilon > 0$, lies in the configuration space. By first order approximation, we have

$$\Psi(p' + \epsilon \Delta p) - \Psi(p') \approx -\epsilon \sum_{i=1}^{N-1} |\Delta \vec{x}_i|^2 < 0 \quad (4.38)$$

This in particular shows that p' is unstable. \square

The idea behind the decentralized control law is clear: each controller \vec{x}_i , $i \neq 1$, has a local potential

$$\Psi_{1i}(\vec{x}_1, \vec{x}_i) := \frac{1}{2}(|\vec{x}_1 - \hat{x}_1|^2 + |\vec{x}_i - \hat{x}_i|^2) \quad (4.39)$$

concerning only himself and agent \vec{x}_1 , and the control law \tilde{u}_{1i} is designed to minimize Ψ_{1i} by following differential equation (4.34). If each agent \vec{x}_i , $i \neq 1$, follows the control rule, then the model is a gradient system and the target configuration is the only stable equilibrium associated with the potential function Ψ .

We end this section with a discussion about the uniqueness of the stable equilibrium. In the statement of theorem 4.2, we assume that each agent \vec{x}_i , $i \neq 1$, knows the target position of himself and the target position of agent \vec{x}_1 . This is a key assumption for the decentralized control law to have a unique stable equilibrium. As a contrast, we refer readers to works [2, 7, 8, 11, 26, 30–32], in any of these formation control models, each agent only knows prescribed relative distances between himself and others. This often leads to the existence of multiple stable equilibria, see for example, [2, 7, 8] for problems about counting equilibria in a RMA system. We here also note some work by Ali Belabbas, the author considered in [31, 32] a nonreciprocal formation control model whereby agents take control of their own motions instead of manipulating interactions between them, and he showed in [31] that there is no decentralized feedback control that globally stabilizes the so-called two-cycles formation.

Conclusion

In this thesis, we have investigated the class of reciprocal multi-agent (RMA) system from four perspectives including the analysis of swarm aggregation, the index/co-index formula, the analysis of the genericity of equivariant Morse functions, and the controllability of a related formation control model. In this section, we discuss possible directions for further development of results we established in this thesis. The discussion is divided into four parts.

1. *Swarm aggregation.* In chapter I, we have assumed that each interaction function g_{ij} satisfies conditions of *strong repulsion* to establish the existence of the solution, i.e, there is no collision of adjacent agents along the gradient flow. However, in the design and control of unmanned autonomous vehicles (UAV's) or multi-robot systems, strong repulsion may not be realizable. We then ask whether the property of collision-free still holds if we replace the condition of *strong repulsion* by *finite repulsion*, i.e, we assume $-\infty < \lim_{d \rightarrow 0} g_{ij}(d) < 0$ for each edge ij in graph Γ . In some of my works not included in this thesis, I have investigated this question under the assumption that Γ is a complete graph, and proved that the main theorem of chapter I will still be true even if we assume the condition of *finite repulsion*. And we expect to generalize the result to an arbitrary connected graph. We also notice that in some works about multi-agent systems, interactions between agents are assumed to be short-ranged. This often happens in the situation where agents have finite scopes or they have their own purposes of interests. The graph connectivity then varies along the evolution, and can be used, for example, to model certain natural phenomena such as formation reconfiguration [16], explaining why and how swarms merge or split. Research is needed to understand the flow behavior, and the tool of clustering we developed in chapter I may be useful.

2. *The index/co-index formula.* In chapter II, we have assumed that the network topology Γ of a RMA system is a Laman graph of type-I(LGT-I), and we established the index/co-index formula. One direct application of the formula is to locate or place a family of *strongly-nondegenerate* critical orbits of any Morse indices/co-indices. A further development is then to apply the formula to compute the Euler characteristic of the configuration space by choosing a particular family of interaction laws. In fact, the index/co-index formula is only one of the many results we can get by assuming that Γ is a LGT-I, there are more properties we can discover from this particular class of RMA systems.

Some of my work suggests that the potential functions associated with this class of RMA systems are generically equivariant Morse functions. Also, if we assume that each interaction function g_{ij} is monotonically increasing and has only one zero, then all stable critical formations are *strongly-nondegenerate*. But so far, we have only focussed on the two dimensional case, research is needed to set up parallel versions of LGT-I for higher dimensional cases, and to generalize the index/co-index formula for these cases.

3. Genericity of equivariant Morse functions. In chapter III, we proved that on a closed space of nondegenerate configurations, the potential function associated with a RMA system with a complete graph is generically an equivariant Morse function, but in this thesis we haven't verified or disproven the genericity assumption on degenerate configurations. We here note some related results in lower dimensional cases. We first notice that in one dimensional case, the genericity assumption actually holds on the entire configuration space because there is no configuration with collision of agents, so there is no degenerate line configuration. Some of my work suggests that the genericity assumption holds in both $2D$ and $3D$ cases. Research is needed to understand higher dimensional cases. We also note that in this thesis, we have only considered the case where the network topology is a complete graph, so the open question whether the genericity assumption holds for an arbitrary rigid graph still remains unanswered

4. Formation Control. We have seen that the control model we considered in chapter IV can be interpreted in different ways in terms of patterns of information flow. This flexibility, together with the main results we established in chapter IV, makes it possible for us to design various control structures for the model. Here is an example by adding a structure of hierarchy, and we illustrate the idea in figure 15. Suppose each boss/sub-boss has the coordinates of its employees (including himself) at any time, and knows the target positions of its employees. Then some of my work suggests that we can design a local potential function for each boss/sub-boss which measures how far his employees are away from the target positions. By approximate path-controllability, each boss/sub-boss is able to steer the subsystem to approximate the gradient flow associated with his own local potential. If all agents obey this decentralized rule, then they work together to minimize a global potential function

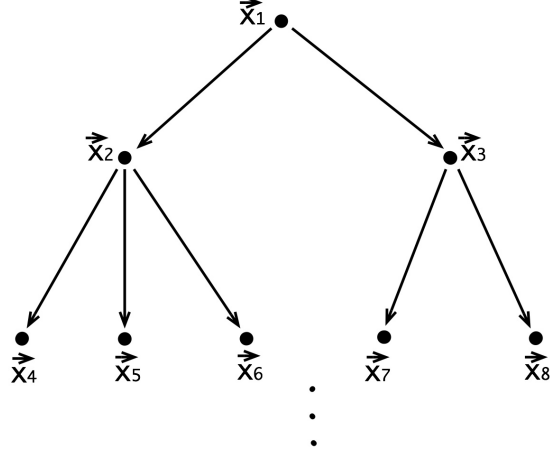


Figure 15: The graph in this figure describes a particular pattern of information flow. There is a unique boss \vec{x}_1 , it takes control of interactions between himself and all level-1 sub-bosses, and in this figure, they are \vec{x}_2 and \vec{x}_3 . While each level-1 sub-boss takes control of interactions between himself and level-2 sub-bosses, and so on so forth.

that has a unique stable equilibrium as the target configuration. But research is needed to investigate the robustness of this decentralized control law because each boss/sub-boss can not follow exactly the gradient flow of its local potential, but only approximate the trajectory. We need to know whether or not this discrepancy will result in the failure of the convergence of the whole system to the target configuration.

We here also note some other possible directions related to this work. For example, if various disturbances need to be taken into account in modeling swarm behaviors or designing multi-robot systems, then the analysis of a stochastic version of the class of RMA systems is needed. We also notice that in some cases, the underlying space of agents may not be Euclidean space, for example, spacecrafts/satellites are evolving on the earth, or we consider the situation where each agent represents an orthonormal frame and the goal is to achieve consensus, then the underlying space is naturally the special orthogonal group. So a parallel version of RMA systems on Riemannian manifolds is needed. Finally, if we are designing a real-time multi-agent system, then communications between agents are important, questions about time delay, channel noise, packet dropouts, limited bit rates and etc. (see,

for example [34] about various challenges and problems in real network system) are all important factors that we should take into account.

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